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# Composition in $\mathcal{OPRA}_m$

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## 1 Introduction

The Oriented Point Relation Algebra $\mathcal{OPRA}_m$, introduced in Moratz (2004), is a binary orientation calculus in the domain of oriented points (o-points) with adjustable granularity. Because of the variable granularity of the calculus, a tabular based representation of the composition operation is infeasible. Two sketches of algorithms for composition have been given (Moratz et al., 2005; Moratz, 2006), but none of these is complete and error-free. In this paper, we will provide a complete $O(m^2)$-time composition algorithm for $\mathcal{OPRA}_m$, which is based on the approach presented in Moratz (2006) and extended to cope with the so-called "same relations".

After a very brief overview on the $\mathcal{OPRA}_m$ calculus in Section 2, Section 3 presents the detailed composition algorithm. Properties of the composition operation, in particular algebraic closure and closure under constraints, will be discussed in Section 4.

# 2 $\mathcal{OPRA}_m$

The domain of $\mathcal{OPRA}_m$ is the domain of oriented points (o-points) in the plane. An o-point $\vec{A}$ in the plane is defined as $\vec{A} = (A, \theta)$ with a point $A \in \mathbb{R}^2$ and an angle $\theta \in \mathbb{R},\ 0 \leq \theta < 2\pi$. Whenever we will refer to o-points within this work, it will be o-points in the plane. Let two o-points $\vec{A} = (A, \theta_A)$ and $\vec{B} = (B, \theta_B)$ be given. An $\mathcal{OPRA}_m$ relation $\vec{A}\ rel_{AB}\ \vec{B}$ describes the relative position of the o-points towards each other.

In $\mathcal{OPRA}_m$, $m$ straight lines partition the plane around an o-point into $2m$ planar and $2m$ linear regions[1]. Each region is assigned a number $i \in \mathcal{Z}_{4m}$[2], starting with 0 for the linear sector that matches the direction of the o-point.

If $A \neq B$ the relation $\vec{A}\ {}_m\angle_i^j\ \vec{B}$ ($m \in \mathbb{N}$, $i, j \in \mathcal{Z}_{4m}$) reads like this: Given a granularity $m$, the relative position of $\vec{B}$ with respect to $\vec{A}$ is described by $i$ (i.e.: $\vec{B}$ lies in sector $i$ of o-point $\vec{A}$) and the relative position of $\vec{A}$ with respect to $\vec{B}$ is described by $j$. This results in $4m \cdot 4m$ possible relations $\vec{A}\ {}_m\angle_i^j\ \vec{B}$. The angle $\varphi_{AB}$ stands for the direction of the straight line $\overline{AB}$ with respect to the direction of $A$, $\theta_A$. So for $\vec{A} = (A, \theta_A)$, $\vec{B} = (B, \theta_B)$, $\vec{A}\ {}_m\angle_i^j\ \vec{B}$ we have

$$\begin{aligned}
&\Bigg(\bigg((i \equiv_2 1) \wedge \left(2\pi\frac{i-1}{4m} < \varphi_{AB} - \theta_A < 2\pi\frac{i+1}{4m}\right)\bigg)\\
&\qquad \vee \bigg((i \equiv_2 0) \wedge \left(\varphi_{AB} - \theta_A = 2\pi\frac{i}{4m}\right)\bigg)\Bigg)\\
\wedge\ &\Bigg(\bigg((j \equiv_2 1) \wedge \left(2\pi\frac{j-1}{4m} < \varphi_{AB} - \theta_B < 2\pi\frac{j+1}{4m}\right)\bigg)\\
&\qquad \vee \bigg((j \equiv_2 0) \wedge \left(\varphi_{AB} - \theta_B = 2\pi\frac{j}{4m}\right)\bigg)\Bigg)
\end{aligned}$$

Figure 1 shows an example for $\mathcal{OPRA}_2$ and $\mathcal{OPRA}_4$.

If $A = B$, the relation $\vec{A}\ {}_m\angle i\ \vec{B}$ denotes which sector $i$ of $\vec{A}$ contains the direction of $\vec{B}$, $\theta_B$. These $4m$ base relations are called *"same" relations*. Thus, the set $\mathcal{BR}$ of base relations of $\mathcal{OPRA}_m$ is $\mathcal{BR} = \{\ {}_m\angle_i^j\ | i, j \in \mathcal{Z}_{4m}\} \cup \{\ {}_m\angle i\ | i \in \mathcal{Z}_{4m}\}$; $|\mathcal{BR}| = 4m(4m+1)$. For "same" relations $\vec{A}\ {}_m\angle i\ \vec{B}$ ($\vec{A} = (A, \theta_A)$, $\vec{B} = (B, \theta_B)$) holds

$$\left((i \equiv_2 0) \wedge \left(\theta_B - \theta_A = 2\pi\frac{i}{4m}\right)\right) \vee \left((i \equiv_2 1) \wedge \left(2\pi\frac{i-1}{4m} < \theta_B - \theta_A < 2\pi\frac{i+1}{4m}\right)\right)$$

See Moratz (2006) for more details on $\mathcal{OPRA}_m$ relations.

[1] Linear regions are also referred to as "linear sectors" within this work and the given references.

[2] $\mathcal{Z}_{4m}$ denotes a cyclic group with $4m$ elements.

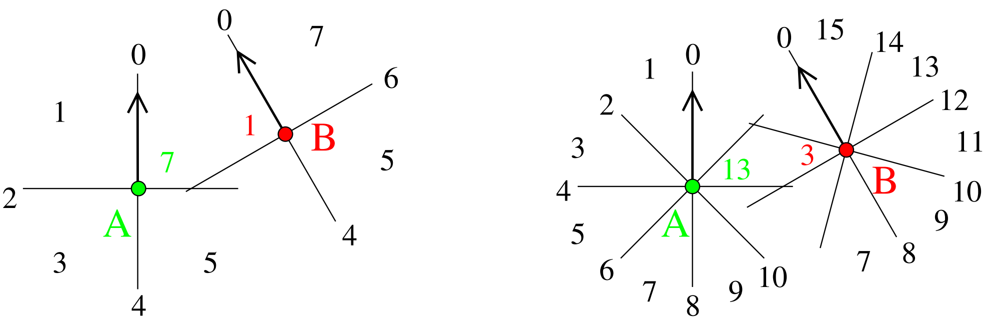


**Figure 1:** Two configurations of o-points $\vec{A}$ and $\vec{B}$ under different instances of $\mathcal{OPRA}_m$. Left: $\vec{A}\ {}_2\angle_7^1\ \vec{B}$, right: $\vec{A}\ {}_4\angle_{13}^3\ \vec{B}$

# 3 A composition algorithm for $\mathcal{OPRA}_m$

Let three arbitrary o-points $\vec{A} = (A, \theta_A)$, $\vec{B} = (B, \theta_B)$, and $\vec{C} = (C, \theta_C)$, and two $\mathcal{OPRA}_m$ relations $\vec{A}\ rel_{AB}\ \vec{B}$ and $\vec{B}\ rel_{BC}\ \vec{C}$ $(rel_{AB}, rel_{BC} \in \mathcal{BR})$ be given. The result of the composition operation $rel_{AB} \circ rel_{BC}$ is the set of all relations $rel_{CA} \in \mathcal{BR}$ between $\vec{C}$ and $\vec{A}$ so that $\vec{A}\ rel_{AB}\ \vec{B} \wedge \vec{B}\ rel_{BC}\ \vec{C} \wedge \vec{C}\ rel_{CA}\ \vec{A}$ has a consistent realization in $\mathbb{R}^2$.

In this section, we first identify 10 different classes of configurations of three o-points for composition operations without "same" relations (Section 3.1), four classes for cases with only "same" relation (Section 3.2), and also four classes of configurations with exactly one "same" relation (Section 3.3). Using this classes, a complete algorithm to calculate the result of the composition operation can be given in Section 3.4.

## 3.1 Composition without "same" relations

If $\vec{A}, \vec{B}$, and $\vec{C}$ are pairwise unequal, the three o-points build a triangle, i.e., the angles $\alpha = \varphi_{AB}$, $\beta = \varphi_{BC}$, and $\gamma = \varphi_{CA}$ sum up to a half circle: $\alpha + \beta + \gamma = \pi$ (see Fig. 2). The triangle may be degenerate. Without loss of generality (w.l.o.g.), we assume it to be positively oriented; for the negatively oriented case, see Section 3.1.11. The $\mathcal{OPRA}_m$ relations are

$$rel_{AB} = {}_m\angle_i^j\ , \quad rel_{BC} = {}_m\angle_k^l\ , \quad rel_{CA} = {}_m\angle_s^t$$

The variables $i, j, k, l, s, t$ describe the configuration of the three o-points, which can be written as a matrix $\binom{t\,j\,l}{i\,k\,s} \in \mathbb{Z}_{4m}^{3\times 2}$. $i$ and $t$ denote sectors of o-point $\vec{A}$, $j$ and $k$ of $\vec{B}$, and $l$ and $s$ of $\vec{C}$.

The properties of this triangle depend on whether $i, j, k, l, s, t$ are linear "sectors" or not, i.e., whether $i \equiv_2 0$ or $i \equiv_2 1$ (and analogously for the other variables). To classify these configurations, we use the canonical transformation $\Psi : \mathbb{Z}_{4m}^{3\times 2} \to \mathbb{Z}_2^{3\times 2}$:

$$\Psi \begin{pmatrix} t & j & l \\ i & k & s \end{pmatrix} = \begin{pmatrix} t \bmod 2 & j \bmod 2 & l \bmod 2 \\ i \bmod 2 & k \bmod 2 & s \bmod 2 \end{pmatrix}$$

A 0 in the matrix denotes a linear sector in an $\mathcal{OPRA}_m$ relation between two o-points.

We can now identify 10 different classes $\mathscr{K}_1, \ldots, \mathscr{K}_{10}$ of configurations for $A = \Psi(\binom{t\,j\,l}{i\,k\,s})$:

$$\mathscr{K}_1 = \left\{\tbinom{000}{000}\right\}$$

$$\mathscr{K}_2 = \left\{\tbinom{100}{000}, \tbinom{000}{100}, \tbinom{010}{000}, \tbinom{000}{010}, \tbinom{001}{000}, \tbinom{000}{001}\right\}$$

$$\begin{aligned}\mathscr{K}_3 &= \left\{\tbinom{110}{000}, \tbinom{101}{000}, \tbinom{011}{000}, \tbinom{000}{110}, \tbinom{000}{101}, \tbinom{000}{011}, \tbinom{100}{010}, \ldots\right\} \\ &= \{A \in \mathbb{Z}_2^{3\times 2} \mid \text{ two column vectors of } A \text{ are unit vectors, one a zero vector}\}\end{aligned}$$

$$\begin{aligned}\mathscr{K}_4 &= \left\{\tbinom{111}{000}, \tbinom{110}{001}, \tbinom{101}{010}, \tbinom{100}{011}, \tbinom{011}{100}, \ldots\right\} \\ &= \left\{A \in \mathbb{Z}_2^{3\times 2} \mid \text{ all column vectors of } A \text{ are unit vectors}\right\}\end{aligned}$$

$$\mathscr{K}_5 = \left\{\tbinom{100}{100}, \tbinom{010}{010}, \tbinom{001}{001}\right\}$$

$$\begin{aligned}\mathscr{K}_6 &= \left\{\tbinom{110}{100}, \tbinom{100}{110}, \tbinom{101}{100}, \tbinom{100}{101}, \tbinom{110}{010}, \ldots\right\} \\ &= \{A \in \mathbb{Z}_2^{3\times 2} \mid \text{column vectors are a unit vector, a zero vector, and } \tbinom{1}{1}\}\end{aligned}$$

$$\begin{aligned}\mathscr{K}_7 &= \left\{\tbinom{111}{100}, \tbinom{110}{101}, \tbinom{101}{110}, \tbinom{100}{111}, \tbinom{111}{010}, \ldots\right\} \\ &= \{A \in \mathbb{Z}_2^{3\times 2} \mid \text{two column vectors of } A \text{ are unit vectors, one is } \tbinom{1}{1}\}\end{aligned}$$

$$\mathscr{K}_8 = \left\{\tbinom{110}{110}, \tbinom{011}{011}, \tbinom{101}{101}\right\}$$

$$\mathscr{K}_9 = \left\{\tbinom{111}{011}, \tbinom{011}{111}, \tbinom{111}{101}, \tbinom{101}{111}, \tbinom{111}{110}, \tbinom{110}{111}\right\}$$

$$\mathscr{K}_{10} = \left\{\tbinom{111}{111}\right\}$$

See Figure 1 for an example. In the following, we specify the set $K_n$ of valid triangle

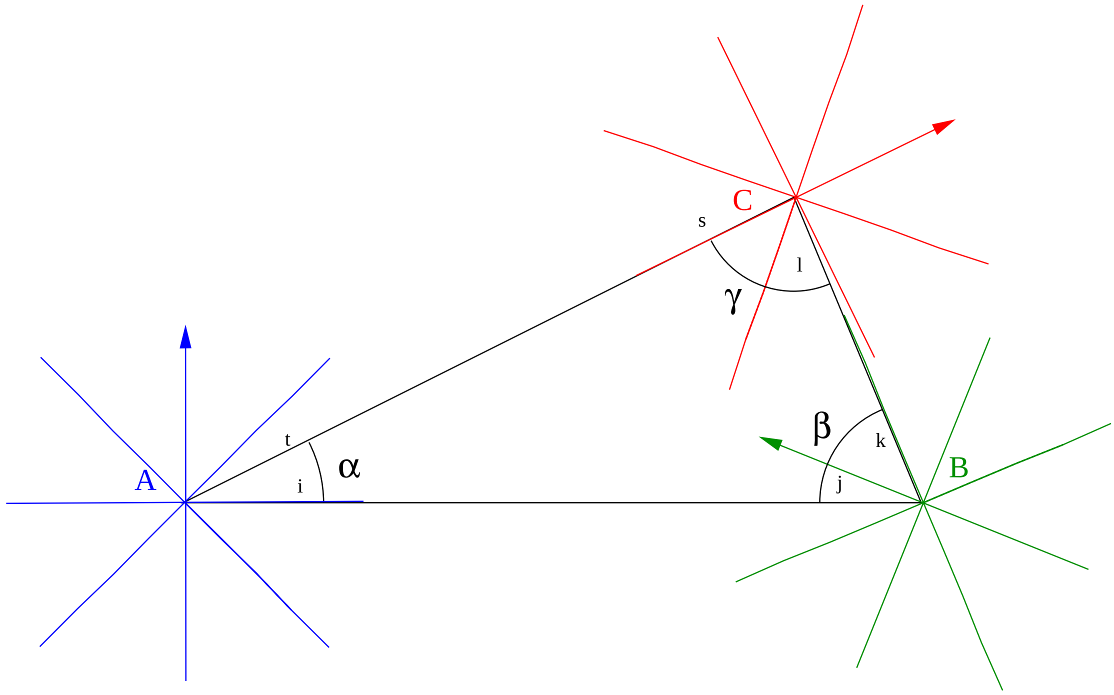


**Figure 2:** An example in $\mathcal{OPRA}_4$: Three o-points in configuration $A = \binom{t\,j\,l}{i\,k\,s} = \binom{13\ \ 1\ \ 11}{12\ 14\ \ 8}$. The corresponding configuration class is $\Psi(A) = \binom{1\,1\,1}{0\,0\,0}$.

configurations for every configuration class $\mathscr{K}_n$, using the fact that the inner angles of the triangle sum up to $\pi$. The binary operator $\ominus$ denotes subtraction in $\mathcal{Z}_{4m}$.

### 3.1.1 $\mathscr{K}_1$:

We assume the triangle not to be degenerate, i.e., $(t\ominus i), (j\ominus k), (l\ominus s) \neq 0$. The following holds:

$$\begin{aligned}
&\alpha = (t\ominus i)\frac{\pi}{2m}, \quad \beta = (j\ominus k)\frac{\pi}{2m}, \quad \gamma = (l\ominus s)\frac{\pi}{2m} \\
&\pi = \alpha+\beta+\gamma = ((t\ominus i)+(j\ominus k)+(l\ominus s))\frac{\pi}{2m} \\
&\quad \Rightarrow \quad ((t\ominus i)+(j\ominus k)+(l\ominus s)) = 2m \\
&\quad \Rightarrow K_1{}' = \Big\{\binom{t\,j\,l}{i\,k\,s} \in \mathbb{Z}_{4m}^{3\times 2} \mid \Psi\binom{t\,j\,l}{i\,k\,s} \in \mathscr{K}_1,\ (t\ominus i),(j\ominus k),(l\ominus s) \neq 0 \\
&\qquad\qquad \text{and } (t\ominus i)+(j\ominus k)+(l\ominus s) = 2m\Big\}
\end{aligned}$$

If the triangle is degenerate (one of $(t\ominus i), (j\ominus k), (l\ominus s) = 0$), the following holds:

$$t\ominus i = 0 \Rightarrow \alpha = 0 \Rightarrow \beta+\gamma = \pi$$

$$\Rightarrow (\beta, \gamma) = (\pi, 0) \text{ or } (\beta, \gamma) = (0, \pi)$$

$$\Rightarrow (j \ominus k,\ l \ominus s) = (2m, 0) \text{ or } (j \ominus k,\ l \ominus s) = (0, 2m) \tag{1}$$

Analogously this holds for $(j \ominus k), (l \ominus s) = 0$. This results in

$$K_1'' = \big\{ \left(\begin{smallmatrix} t\,j\,l \\ i\,k\,s \end{smallmatrix}\right) \in \mathbb{Z}_{4m}^{3\times 2} \mid \Psi\left(\begin{smallmatrix} t\,j\,l \\ i\,k\,s \end{smallmatrix}\right) \in \mathscr{K}_1,$$
$$(t \ominus i),\ (j \ominus k),\ (l \ominus s) \in \{(0,0,2m),\ (0,2m,0),\ (2m,0,0)\} \big\}$$

$$K_1 = K_1' \cup K_1''$$

### 3.1.2 $\mathscr{K}_2$:

W.l.o.g. we look at the configuration $\left(\begin{smallmatrix} 100 \\ 000 \end{smallmatrix}\right) \in \mathscr{K}_2$. It holds:

$$\alpha \in \left]((t \ominus i) - 1)\frac{\pi}{2m},\ ((t \ominus i) + 1)\frac{\pi}{2m}\right[ \ ; \quad \beta = (j \ominus k)\frac{\pi}{2m}\ ; \quad \gamma = (l \ominus s)\frac{\pi}{2m}$$

$$\begin{aligned}
&\pi = \alpha + \beta + \gamma \\
&\Rightarrow\ \pi \in \left]((t \ominus i) - 1 + (j \ominus k) + (l \ominus s))\frac{\pi}{2m}\ ,\ ((t \ominus i) + 1 + (j \ominus k) + (l \ominus s))\frac{\pi}{2m}\right[ \qquad (2) \\
&\Rightarrow\ ((t \ominus i) - 1 + (j \ominus k) + (l \ominus s))\frac{\pi}{2m}\ <\ \pi\ <\ ((t \ominus i) + 1 + (j \ominus k) + (l \ominus s))\frac{\pi}{2m} \\
&\Rightarrow\ (t \ominus i) - 1 + (j \ominus k) + (l \ominus s)\ <\ 2m\ <\ (t \ominus i) + 1 + (j \ominus k) + (l \ominus s) \\
&\Rightarrow\ -1\ <\ 2m - ((t \ominus i) + (j \ominus k) + (l \ominus s))\ <\ 1 \\
&\Rightarrow\ -2m - 1\ <\ -((t \ominus i) + (j \ominus k) + (l \ominus s))\ <\ -2m + 1 \\
&\Rightarrow\ 2m - 1\ <\ (t \ominus i) + (j \ominus k) + (l \ominus s)\ <\ 2m + 1 \\
&\Rightarrow\ (t \ominus i) + (j \ominus k) + (l \ominus s) = 2m, \quad \text{because } (t \ominus i) + (j \ominus k) + (l \ominus s) \in \mathbb{Z} \quad \lightning
\end{aligned}$$

This is a contradiction, because following the definition, $i + j + k + l + s + t \equiv_2 1$ for all configurations in $\mathscr{K}_2$. As (2) is the same for all other representatives of $\mathscr{K}_2$, there is no valid triangle configuration in $\mathscr{K}_2$: $K_2 = \emptyset$. In the following, we will also just consider one representative.

### 3.1.3 $\mathscr{K}_3$:

In $\mathscr{K}_3$ there is exactly on pair of variables (w.l.o.g. $l$ and $s$) which denotes two linear sectors. Let the triangle be not degenerate, i.e., $l \neq s$. It holds:

$$\alpha \in \left]((t \ominus i) - 1)\frac{\pi}{2m}\ ,\ ((t \ominus i) + 1)\frac{\pi}{2m}\right[$$

$$\beta \in \left]((j \ominus k) - 1)\frac{\pi}{2m}\ ,\ ((j \ominus k) + 1)\frac{\pi}{2m}\right[$$
$$\gamma = (l \ominus s)\frac{\pi}{2m}$$

$$\begin{aligned}
\pi =& \alpha + \beta + \gamma \\
\Rightarrow\ & \pi \in \left]((t \ominus i) + (j \ominus k) - 2 + (l \ominus s))\frac{\pi}{2m}, ((t \ominus i) + (j \ominus k) + 2 + (l \ominus s))\frac{\pi}{2m}\right[ \\
\Rightarrow\ & ((t \ominus i) + (j \ominus k) + (l \ominus s) - 2)\frac{\pi}{2m}\ <\ \pi\ <\ ((t \ominus i) + (j \ominus k) + (l \ominus s) + 2)\frac{\pi}{2m} \\
\Rightarrow\ & (t \ominus i) + (j \ominus k) + (l \ominus s) - 2\ <\ 2m\ <\ (t \ominus i) + (j \ominus k) + (l \ominus s) + 2 \\
\Rightarrow\ & 2m - 2\ <\ (t \ominus i) + (j \ominus k) + (l \ominus s)\ <\ 2m + 2 \\
\Rightarrow\ & (t \ominus i) + (j \ominus k) + (l \ominus s) = 2m, \quad \text{because } (i + j + k + l + s + t) \equiv_2 0
\end{aligned}$$

$$\Rightarrow {K_3}' = \left\{ \left(\begin{smallmatrix} t\,j\,l \\ i\,k\,s \end{smallmatrix}\right) \in \mathbb{Z}_{4m}^{3\times 2} \mid \Psi\left(\begin{smallmatrix} t\,j\,l \\ i\,k\,s \end{smallmatrix}\right) \in \mathscr{K}_3, \right.$$
$$\left. t \neq i, j \neq k, l \neq s \text{ and } (t \ominus i) + (j \ominus k) + (l \ominus s) = 2m \right\}$$

In the case of a degenerate triangle, $l \ominus s = 0$.

$$\begin{aligned}
& l \ominus s = 0 \quad \Rightarrow \quad \gamma = 0 \quad \Rightarrow \quad \alpha + \beta = \pi \\
\Rightarrow\ & (\alpha, \beta) = (\pi, 0) \text{ or } (\alpha, \beta) = (0, \pi) \\
\Rightarrow\ & (t \ominus i, j \ominus k) \in \{(2m, 0), (0, 2m)\} \quad ⚡
\end{aligned} \tag{3}$$

This is a contradiction, because $(t \ominus i) \equiv_2 (j \ominus k) \equiv_2 1$. No configuration in $\mathscr{K}_3$ can form a degenerate triangle, therefore holds:

$$K_3 = {K_3}'$$

### 3.1.4 $\mathscr{K}_4$:

W.l.o.g. we look at the configuration $\left(\begin{smallmatrix} 111 \\ 000 \end{smallmatrix}\right)$. It holds:

$$\alpha \in \left]((t \ominus i) - 1)\frac{\pi}{2m}\ ,\ ((t \ominus i) + 1)\frac{\pi}{2m}\right[$$
$$\beta \in \left]((j \ominus k) - 1)\frac{\pi}{2m}\ ,\ ((j \ominus k) + 1)\frac{\pi}{2m}\right[$$
$$\gamma \in \left]((l \ominus s) - 1)\frac{\pi}{2m}\ ,\ ((l \ominus s) + 1)\frac{\pi}{2m}\right[$$

$$\pi = \alpha + \beta + \gamma$$

$$\Rightarrow\ \pi \in \left]((t\ominus i)+(j\ominus k)-3+(l\ominus s))\frac{\pi}{2m},\quad ((t\ominus i)+(j\ominus k)+(l\ominus s)+3)\frac{\pi}{2m}\right[$$
$$\Rightarrow\ ((t\ominus i)+(j\ominus k)+(l\ominus s)-3)\frac{\pi}{2m}\ <\ \pi\ <\ ((t\ominus i)+(j\ominus k)+(l\ominus s)+3)\frac{\pi}{2m}$$
$$\Rightarrow\ (t\ominus i)+(j\ominus k)+(l\ominus s)-3\ <\ 2m\ <\ (t\ominus i)+(j\ominus k)+(l\ominus s)+3$$
$$\Rightarrow\ 2m-3\ <\ (t\ominus i)+(j\ominus k)+(l\ominus s)\ <\ 2m+3$$
$$\Rightarrow\ (t\ominus i)+(j\ominus k)+(l\ominus s)\in\{2m-2,\ 2m-1,\ 2m,\ 2m+1,\ 2m+2\}$$

Furthermore holds:

$$i+j+k+l+s+t\equiv_2 1$$
$$\Rightarrow\ (t\ominus i)+(j\ominus k)+(l\ominus s)\in\{2m-1,2m+1\}$$
$$\Rightarrow\ K_4=\left\{\left(\begin{smallmatrix}t\,j\,l\\ i\,k\,s\end{smallmatrix}\right)\in\mathbb{Z}_{4m}^{3\times 2}\mid \Psi\left(\begin{smallmatrix}t\,j\,l\\ i\,k\,s\end{smallmatrix}\right)\in\mathscr{K}_4\ ,\ (t\ominus i)+(j\ominus k)+(l\ominus s)\in\{2m-1,2m+1\}\right\}$$

### 3.1.5 $\mathscr{K}_5$:

W.l.o.g. we look at the configuration $\left(\begin{smallmatrix}100\\ 100\end{smallmatrix}\right)$.

**Case 1:** Let $(t\ominus i)$, $(j\ominus k)$, $(l\ominus s)\neq 0$. If follows that $t\ominus i\geq 2$, and it holds:

$$\alpha\in\left]((t\ominus i)-2)\frac{\pi}{2m},\ ((t\ominus i)+2)\frac{\pi}{2m}\right[\quad;\quad \beta=(j\ominus k)\frac{\pi}{2m}\quad;\quad \gamma=(l\ominus s)\frac{\pi}{2m}$$

$$\pi=\alpha+\beta+\gamma$$
$$\Rightarrow\ \pi\in\left]((t\ominus i)-2+(j\ominus k)+(l\ominus s))\frac{\pi}{2m},\quad ((t\ominus i)+2+(j\ominus k)+(l\ominus s))\frac{\pi}{2m}\right[$$
$$\Rightarrow\ 2m-2<(t\ominus i)+(j\ominus k)+(l\ominus s)<2m+2$$
$$\Rightarrow\quad (t\ominus i)+(j\ominus k)+(l\ominus s)=2m,\quad \text{because } i+j+k+l+s+t\equiv_2 0$$

The same holds for $\left(\begin{smallmatrix}010\\ 010\end{smallmatrix}\right)$ with $j\ominus k\neq 0$ and $\left(\begin{smallmatrix}001\\ 001\end{smallmatrix}\right)$ with $l\ominus s\neq 0$.

$$\Rightarrow K_5{}'=\left\{\left(\begin{smallmatrix}t\,j\,l\\ i\,k\,s\end{smallmatrix}\right)\in\mathbb{Z}_{4m}^{3\times 2}\mid \Psi\left(\begin{smallmatrix}t\,j\,l\\ i\,k\,s\end{smallmatrix}\right)\in\mathscr{K}_5,\ (t\ominus i)+(j\ominus k)+(l\ominus s)=2m\ ,\ (t\ominus i),(j\ominus k),(l\ominus s)\neq 0\right\}$$

**Case 2:** Let one of the three variable pairs consist of equal values. If $t\ominus i=0$, it holds:

$$\alpha\in\left[0,\ \frac{\pi}{m}\right[\quad;\quad \beta=(j\ominus k)\frac{\pi}{2m}\quad;\quad \gamma=(l\ominus s)\frac{\pi}{2m}$$

$$\pi=\alpha+\beta+\gamma$$
$$\Rightarrow\ \pi\in\left[((j\ominus k)+(l\ominus s))\frac{\pi}{2m},\quad (2+(j\ominus k)+(l\ominus s))\frac{\pi}{2m}\right[$$

$$\begin{aligned}
&\Rightarrow\ ((j \ominus k) + (l \ominus s))\frac{\pi}{2m}\ \le\ \pi\ <\ (2 + (j \ominus k) + (l \ominus s))\frac{\pi}{2m}\\
&\Rightarrow\ (j \ominus k) + (l \ominus s)\ \le\ 2m\ <\ 2 + (j \ominus k) + (l \ominus s)\\
&\Rightarrow\ 2m - 2\ <\ (j \ominus k) + (l \ominus s)\ \le\ 2m\\
&\Rightarrow\ (j \ominus k) + (l \ominus s) = 2m, \quad \text{because } j + k + l + s \equiv_2 0\\
&\Rightarrow\ \beta + \gamma = ((j \ominus k) + (l \ominus s))\frac{\pi}{2m} = \pi\\
&\Rightarrow\ \alpha = 0\\
&\Rightarrow\ \text{the triangle is degenerate: } (\alpha, \beta, \gamma) \in \{(0, \pi, 0),\ (0, 0, \pi)\}\\
&\Rightarrow\ ((t \ominus i), (j \ominus k), (l \ominus s)) \in \{(0, 2m, 0),\ (0, 0, 2m)\}
\end{aligned}$$

If $j \ominus k = 0$ or $l \ominus s = 0$, equation (1) holds, and we get:

$$((t \ominus i), (j \ominus k), (l \ominus s)) \in \{(2m, 0, 0),\ (0, 0, 2m)\}$$
$$\text{or } ((t \ominus i), (j \ominus k), (l \ominus s)) \in \{(0, 2m, 0),\ (2m, 0, 0)\}, \text{ respectively}$$

$$\Rightarrow\ K_5'' := \Big\{ \left(\begin{smallmatrix} t\,j\,l \\ i\,k\,s \end{smallmatrix}\right) \in \mathbb{Z}_{4m}^{3\times 2} \mid \Psi\left(\begin{smallmatrix} t\,j\,l \\ i\,k\,s \end{smallmatrix}\right) \in \mathscr{K}_5\ ,\\ ((t \ominus i), (j \ominus k), (l \ominus s)) \in \{(0, 2m, 0),\ (0, 0, 2m),\ (2m, 0, 0)\} \Big\}$$

Summed up, we get

$$K_5 = K_5' \cup K_5''$$

### 3.1.6 $\mathscr{K}_6$:

Let $M = \left(\begin{smallmatrix} a_1\,a_2\,a_3 \\ b_1\,b_2\,b_3 \end{smallmatrix}\right) \in \mathbb{Z}_{4m}^{3\times 2}$ and $M' = \Psi\left(\begin{smallmatrix} a_1\,a_2\,a_3 \\ b_1\,b_2\,b_3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_1'\,a_2'\,a_3' \\ b_1'\,b_2'\,b_3' \end{smallmatrix}\right) \in \mathscr{K}_6$. We can distinguish three cases:

**Case 1:** Let $x \neq y$, $a_x' = b_x' = 1, a_y' = b_y' = 0$ and both $a_x \ominus b_x \neq 0, a_y \ominus b_y \neq 0$. W.l.o.g. $M' = \left(\begin{smallmatrix} 110 \\ 100 \end{smallmatrix}\right)$, $a_x = t, b_x = i, a_y = l, b_y = s$. It holds:

$$\alpha \in \Big]((t \ominus i) - 2)\frac{\pi}{2m},\ ((t \ominus i) + 2)\frac{\pi}{2m}\Big[$$
$$\beta \in \Big]((j \ominus k) - 1)\frac{\pi}{2m},\ ((j \ominus k) + 1)\frac{\pi}{2m}\Big[$$
$$\gamma = (l \ominus s)\frac{\pi}{2m}$$

$$\begin{aligned}
&\pi = \alpha + \beta + \gamma\\
&\Rightarrow\ \pi \in \Big]((t \ominus i) + (j \ominus k) - 3 + (l \ominus s))\frac{\pi}{2m},\quad ((t \ominus i) + (j \ominus k) + 3 + (l \ominus s))\frac{\pi}{2m}\Big[\\
&\Rightarrow\ 2m - 3\ <\ (t \ominus i) + (j \ominus k) + (l \ominus s)\ <\ 2m + 3
\end{aligned}$$

Furthermore:

$$i + j + k + l + s + t \equiv 1 \bmod 2$$
$$\Rightarrow (t \ominus i) + (j \ominus k) + (l \ominus s) \in \{2m-1, 2m+1\}$$

$$\Rightarrow {K_6}' = \Big\{ \left(\begin{smallmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{smallmatrix}\right) \in \mathbb{Z}_{4m}^{3\times 2} \mid \Psi\left(\begin{smallmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a'_1 & a'_2 & a'_3 \\ b'_1 & b'_2 & b'_3 \end{smallmatrix}\right) \in \mathscr{K}_6 \;, \\ x \neq y, a'_x = b'_x = 1, a'_y = b'_y = 0, a_x \ominus b_x \neq 0, a_y \ominus b_y \neq 0, \\ \sum_n (a_n \ominus b_n) \in \{2m-1, 2m+1\} \Big\}$$

**Case 2:** Let $a'_x = b'_x = 0$ and $a_x \ominus b_x = 0$. ↯ This is impossible because of a similar contradiction as given in (3), page 7.

**Case 3:** Let $x \neq y$, $a'_x = b'_x = 1, a'_y = b'_y = 0$ and $a_x \ominus b_x = 0, a_y \ominus b_y \neq 0$. W.l.o.g. $M' = \left(\begin{smallmatrix} 110 \\ 100 \end{smallmatrix}\right)$ again, $a_x = t, b_x = i, a_y = l, b_y = s$. It holds:

$$\alpha \in \Big[0, \frac{\pi}{m}\Big[ \quad , \quad \beta \in \Big]((j \ominus k) - 1)\frac{\pi}{2m}, ((j \ominus k) + 1)\frac{\pi}{2m}\Big[ \quad , \quad \gamma = (l \ominus s)\frac{\pi}{2m}$$

$$\begin{aligned} & \pi = \alpha + \beta + \gamma \\ & \Rightarrow \pi \in \Big]((j \ominus k) - 1 + (l \ominus s))\frac{\pi}{2m}, \quad (2 + (j \ominus k) + 1 + (l \ominus s))\frac{\pi}{2m}\Big[ \\ & \Rightarrow ((j \ominus k) - 1 + (l \ominus s))\frac{\pi}{2m} < \pi < (2 + (j \ominus k) + 1 + (l \ominus s))\frac{\pi}{2m} \\ & \Rightarrow (j \ominus k) + (l \ominus s) - 1 < 2m < (j \ominus k) + (l \ominus s) + 3 \\ & \Rightarrow 2m - 3 < (j \ominus k) + (l \ominus s) < 2m + 1 \\ & \Rightarrow (j \ominus k) + (l \ominus s) = 2m - 1, \quad \text{because } j + k + l + s \equiv_2 1 \end{aligned}$$

$$\Rightarrow {K_6}'' = \Big\{ \left(\begin{smallmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{smallmatrix}\right) \in \mathbb{Z}_{4m}^{3\times 2} \mid \Psi\left(\begin{smallmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a'_1 & a'_2 & a'_3 \\ b'_1 & b'_2 & b'_3 \end{smallmatrix}\right) \in \mathscr{K}_6 \;, x \neq y, \\ a'_x = b'_x = 1, a'_y = b'_y = 0, a_x \ominus b_x = 0, a_y \ominus b_y \neq 0, \\ \sum_n (a_n \ominus b_n) = 2m - 1 \Big\}$$

Summed up:

$$K_6 = {K_6}' \cup {K_6}''$$

### 3.1.7 $\mathscr{K}_7$:

Let $M = \binom{a_1\,a_2\,a_3}{b_1\,b_2\,b_3} \in \mathbb{Z}_{4m}^{3\times 2}$ and $M' = \Psi\binom{a_1\,a_2\,a_3}{b_1\,b_2\,b_3} = \binom{a'_1\,a'_2\,a'_3}{b'_1\,b'_2\,b'_3} \in \mathscr{K}_7$. W.l.o.g. $M' = \binom{111}{100}$. We have two cases here:

**Case 1:** Let $a'_x = b'_x = 1$ and $a_x \ominus b_x \neq 0$. W.l.o.g. $a_x = t, b_x = i$.

$$\alpha \in \left]((t \ominus i) - 2)\frac{\pi}{2m},\ ((t \ominus i) + 2)\frac{\pi}{2m}\right[$$
$$\beta \in \left]((j \ominus k) - 1)\frac{\pi}{2m},\ ((j \ominus k) + 1)\frac{\pi}{2m}\right[$$
$$\gamma \in \left]((l \ominus s) - 1)\frac{\pi}{2m},\ ((l \ominus s) + 1)\frac{\pi}{2m}\right[$$

$$\pi = \alpha + \beta + \gamma$$
$$\Rightarrow\ \pi \in \left]((t \ominus i) + (j \ominus k) + (l \ominus s) - 4)\frac{\pi}{2m}\ ,\ ((t \ominus i) + (j \ominus k) + (l \ominus s) + 4)\frac{\pi}{2m}\right[$$
$$\Rightarrow\ 2m - 4\ <\ (t \ominus i) + (j \ominus k) + (l \ominus s)\ <\ 2m + 4$$

Furthermore:

$$i + j + k + l + s + t \equiv_2 0$$
$$\Rightarrow (t \ominus i) + (j \ominus k) + (l \ominus s) \in \{2m - 2, 2m, 2m + 2\}$$

$$\Rightarrow K_7{}' = \Big\{\binom{a_1\,a_2\,a_3}{b_1\,b_2\,b_3} \in \mathbb{Z}_{4m}^{3\times 2} \mid \Psi\binom{a_1\,a_2\,a_3}{b_1\,b_2\,b_3} = \binom{a'_1\,a'_2\,a'_3}{b'_1\,b'_2\,b'_3} \in \mathscr{K}_7\ ,$$
$$a'_x = b'_x = 1, a_x \ominus b_x \neq 0,\quad \sum_n (a_n \ominus b_n) \in \{2m - 2, 2m, 2m + 2\}\Big\}$$

**Case 2:** Let $a'_x = b'_x = 1$ and $a_x \ominus b_x = 0$. W.l.o.g. $a_x = t, b_x = i$.

$$\alpha \in \left[0,\ \frac{\pi}{m}\right[$$
$$\beta \in \left]((j \ominus k) - 1)\frac{\pi}{2m},\ ((j \ominus k) + 1)\frac{\pi}{2m}\right[$$
$$\gamma \in \left]((l \ominus s) - 1)\frac{\pi}{2m},\ ((l \ominus s) + 1)\frac{\pi}{2m}\right[$$

$$\pi = \alpha + \beta + \gamma$$
$$\Rightarrow\ \pi \in \left]((j \ominus k) - 1 + (l \ominus s) - 1)\frac{\pi}{2m},\quad (2 + (j \ominus k) + 1 + (l \ominus s) + 1)\frac{\pi}{2m}\right[$$
$$\Rightarrow\ ((j \ominus k) - 1 + (l \ominus s) - 1)\frac{\pi}{2m}\ <\ \pi\ <\ (2 + (j \ominus k) + 1 + (l \ominus s) + 1)\frac{\pi}{2m}$$
$$\Rightarrow\ (j \ominus k) + (l \ominus s) - 2\ <\ 2m\ <\ (j \ominus k) + (l \ominus s) + 4$$

$$\Rightarrow\ 2m-4 \ < \ (j \ominus k) + (l \ominus s) \ < \ 2m+2$$
$$\Rightarrow\ (j \ominus k) + (l \ominus s) \in \{2m-2,\ 2m\}, \quad \text{because } j+k+l+s \equiv_2 0$$

$$\Rightarrow K_7'' = \Big\{ \left(\begin{smallmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{smallmatrix}\right) \in \mathbb{Z}_{4m}^{3\times 2} \mid \Psi\left(\begin{smallmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a'_1 & a'_2 & a'_3 \\ b'_1 & b'_2 & b'_3 \end{smallmatrix}\right) \in \mathscr{K}_7\ , \\ a'_x = b'_x = 1, a_x \ominus b_x = 0, \quad \sum_n (a_n \ominus b_n) \in \{2m-2, 2m\} \Big\}$$

And summed up:

$$K_7 = K_7' \cup K_7''$$

### 3.1.8 $\mathscr{K}_8$:

Let $M = \left(\begin{smallmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{smallmatrix}\right) \in \mathbb{Z}_{4m}^{3\times 2}$ and $M' = \Psi(M) = \left(\begin{smallmatrix} a'_1 & a'_2 & a'_3 \\ b'_1 & b'_2 & b'_3 \end{smallmatrix}\right) \in \mathscr{K}_8$. W.l.o.g. $M' = \left(\begin{smallmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{smallmatrix}\right)$. We can distinguish three cases:

**Case 1:** Let $a'_x = b'_x = 0$ and $a_x \ominus b_x = 0$. W.l.o.g. $a_x = t, b_x = i$.

$$t \ominus i = 0 \quad \Rightarrow \alpha = 0 \quad \Rightarrow \beta + \gamma = \pi$$
$$\Rightarrow (\beta, \gamma) \in \big\{(\pi, 0), (\beta, \gamma)\big\}$$
$$\Rightarrow (j \ominus k,\ l \ominus s) \in \big\{(2m, 0), (0, 2m)\big\}$$

$$\Rightarrow K_8' = \Big\{ \left(\begin{smallmatrix} t & j & l \\ i & k & s \end{smallmatrix}\right) \in \mathbb{Z}_{4m}^{3\times 2} \mid \Psi\left(\begin{smallmatrix} t & j & l \\ i & k & s \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{smallmatrix}\right), \\ t \ominus i = 0,\ ((t \ominus i), (j \ominus k), (l \ominus s)) \in \{(0, 2m, 0), (0, 0, 2m)\} \Big\} \\ \cup \Big\{ \left(\begin{smallmatrix} t & j & l \\ i & k & s \end{smallmatrix}\right) \in \mathbb{Z}_{4m}^{3\times 2} \mid \Psi\left(\begin{smallmatrix} t & j & l \\ i & k & s \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{smallmatrix}\right), \\ j \ominus k = 0,\ ((t \ominus i), (j \ominus k), (l \ominus s)) \in \{(2m, 0, 0), (0, 0, 2m)\} \Big\} \\ \cup \Big\{ \left(\begin{smallmatrix} t & j & l \\ i & k & s \end{smallmatrix}\right) \in \mathbb{Z}_{4m}^{3\times 2} \mid \Psi\left(\begin{smallmatrix} t & j & l \\ i & k & s \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{smallmatrix}\right), \\ l \ominus s = 0\ , ((t \ominus i), (j \ominus k), (l \ominus s)) \in \{(2m, 0, 0), (0, 2m, 0)\} \Big\}$$

**Case 2a:** Let $a'_x = b'_x = 0$ and $a_x \ominus b_x \neq 0$, $a_y \ominus b_y = a_z \ominus b_z = 0$ ($x, y$ and $z$ pairwise unequal). W.l.o.g. $t \ominus i \neq 0, j \ominus k = 0$ and $l \ominus s = 0$.

$$\alpha = (t \ominus i)\frac{\pi}{zm} \quad ; \quad \beta \in \Big[0,\ \frac{\pi}{m}\Big[ \quad ; \quad \gamma \in \Big[0,\ \frac{\pi}{m}\Big[$$

$$\pi = \alpha + \beta + \gamma$$
$$\Rightarrow\ \pi \in \Big[(t \ominus i)\frac{\pi}{2m}, \quad ((t \ominus i) + 2 + 2)\frac{\pi}{2m}\Big[$$
$$\Rightarrow\ (t \ominus i)\frac{\pi}{2m} \leq \pi \ < \ ((t \ominus i) + 2 + 2)\frac{\pi}{2m}$$

$$
\begin{aligned}
&\Rightarrow\ t \ominus i \le 2m\ <\ t \ominus i + 4\\
&\Rightarrow\ 2m - 4\ <\ t \ominus i \le 2m\\
&\Rightarrow\ t \ominus i \in \{2m-2,\ 2m\}, \quad \text{because } t + i \equiv_2 0
\end{aligned}
$$

**Case 2b:** Let now, in contrast to Case 2a, $a_z \ominus b_z \neq 0$; w.l.o.g. $t \ominus i \neq 0, j \ominus k = 0$ and $l \ominus s \neq 0$. Now holds:

$$\alpha = (t \ominus i)\frac{\pi}{2m} \quad ; \quad \beta \in \left[0,\ \frac{\pi}{m}\right[ \quad ; \quad \gamma \in \left]((l \ominus s) - 2)\frac{\pi}{2m},\ ((l \ominus s) + 2)\frac{\pi}{2m}\right[$$

$$
\begin{aligned}
&\pi = \alpha + \beta + \gamma\\
&\Rightarrow\ \pi \in \left]((t \ominus i) + (l \ominus s) - 2)\frac{\pi}{2m}, \quad ((t \ominus i) + 2 + (l \ominus s) + 2)\frac{\pi}{2m}\right[\\
&\Rightarrow\ ((t \ominus i) + (l \ominus s) - 2)\frac{\pi}{2m}\ <\ \pi\ <\ ((t \ominus i) + 2 + (l \ominus s) + 2)\frac{\pi}{2m}\\
&\Rightarrow\ (t \ominus i) + (l \ominus s) - 2\ <\ 2m\ <\ (t \ominus i) + (l \ominus s) + 4\\
&\Rightarrow\ 2m - 4\ <\ (t \ominus i) + (l \ominus s)\ <\ 2m + 2\\
&\Rightarrow\ (t \ominus i) + (l \ominus s) \in \{2m-2,\ 2m\}, \quad \text{because } t + i + l + s \equiv_2 0
\end{aligned}
$$

From Case 2a and 2b now follows:

$$
\begin{aligned}
\Rightarrow K_8'' = \Big\{ &\tbinom{t\,j\,l}{i\,k\,s} \in \mathbb{Z}_{4m}^{3\times 2} \mid \Psi\tbinom{t\,j\,l}{i\,k\,s} = \tbinom{011}{011},\ t \ominus i \neq 0, j \ominus k = 0 \text{ or } l \ominus s = 0, \text{ and}\\
&((t \ominus i), (j \ominus k), (l \ominus s)) \in \{(2m, 0, 0), (2m-2, 0, 0)\}\Big\}\\
\cup \Big\{ &\tbinom{t\,j\,l}{i\,k\,s} \in \mathbb{Z}_{4m}^{3\times 2} \mid \Psi\tbinom{t\,j\,l}{i\,k\,s} = \tbinom{101}{101},\ j \ominus k \neq 0, t \ominus i = 0 \text{ or } l \ominus s = 0, \text{ and}\\
&((t \ominus i), (j \ominus k), (l \ominus s)) \in \{(0, 2m, 0), (0, 2m-2, 0)\}\Big\}\\
\cup \Big\{ &\tbinom{t\,j\,l}{i\,k\,s} \in \mathbb{Z}_{4m}^{3\times 2} \mid \Psi\tbinom{t\,j\,l}{i\,k\,s} = \tbinom{110}{110},\ l \ominus s \neq 0, t \ominus i = 0 \text{ or } j \ominus k = 0, \text{ and}\\
&((t \ominus i), (j \ominus k), (l \ominus s)) \in \{(0, 0, 2m), (0, 0, 2m-2)\}\Big\}
\end{aligned}
$$

**Case 3:** Let all variables within a pair be not identical, i.e., $t \ominus i,\ j \ominus k,\ l \ominus s \neq 0$.

$$
\begin{aligned}
&\alpha = (t \ominus i)\frac{\pi}{2m}\\
&\beta \in \left]((j \ominus k) - 2)\frac{\pi}{2m},\ ((j \ominus k) + 2)\frac{\pi}{2m}\right[\\
&\gamma \in \left]((l \ominus s) - 2)\frac{\pi}{2m},\ ((l \ominus s) + 2)\frac{\pi}{2m}\right[
\end{aligned}
$$

$$
\begin{aligned}
&\pi = \alpha + \beta + \gamma\\
&\Rightarrow\ \pi \in \left]((t \ominus i) + (j \ominus k) + (l \ominus s) - 4)\frac{\pi}{2m}, \quad ((t \ominus i) + (j \ominus k) + (l \ominus s) + 4)\frac{\pi}{2m}\right[\\
&\Rightarrow\ 2m - 4\ <\ (t \ominus i) + (j \ominus k) + (l \ominus s)\ <\ 2m + 4
\end{aligned}
$$

$$\Rightarrow\ (t\ominus i)+(j\ominus k)+(l\ominus s)\in\{2m-2,\ 2m,\ 2m+2\},\quad (i+j+k+l+s+t\equiv_2 0)$$

$$\Rightarrow K_8{}''' = \big\{\left(\begin{smallmatrix} t\,j\,l\\ i\,k\,s\end{smallmatrix}\right)\in\mathbb{Z}_{4m}^{3\times 2}\mid \Psi\left(\begin{smallmatrix} t\,j\,l\\ i\,k\,s\end{smallmatrix}\right)\in\mathscr{K}_8,\ t\ominus i, j\ominus k, l\ominus s\neq 0 \text{ and } (t\ominus i)+(j\ominus k)+(l\ominus s)\in\{2m-2,2m,2m+2\}\big\}$$

Altogether we get:

$$K_8 = K_8{}'\cup K_8{}''\cup K_8{}'''$$

### 3.1.9 $\mathscr{K}_9$:

Again, $M=\left(\begin{smallmatrix} a_1\,a_2\,a_3\\ b_1\,b_2\,b_3\end{smallmatrix}\right)\in\mathbb{Z}_{4m}^{3\times 2}$ and $M'=\Psi(M)=\left(\begin{smallmatrix} a'_1\,a'_2\,a'_3\\ b'_1\,b'_2\,b'_3\end{smallmatrix}\right)\in\mathscr{K}_9$. W.l.o.g. $M'=\left(\begin{smallmatrix}111\\011\end{smallmatrix}\right)$. We have three cases:

**Case 1:** Let $a'_y=b'_y=a'_z=b'_z=1$, $(x\neq y)$, $a_y\ominus b_y=a_z\ominus b_z=0$. W.l.o.g. $j\ominus k=l\ominus s=0$.

$$\alpha\in\left]((t\ominus i)-1)\frac{\pi}{2m},\ ((t\ominus i)+1)\frac{\pi}{2m}\right[\quad;\quad \beta\in\left[0,\ \frac{\pi}{m}\right[\quad;\quad \gamma\in\left[0,\ \frac{\pi}{m}\right[$$

$$\begin{aligned}&\pi=\alpha+\beta+\gamma\\ &\ \Rightarrow\ \pi\in\left]((t\ominus i)-1)\frac{\pi}{2m},\quad ((t\ominus i)+1+2+2)\frac{\pi}{2m}\right[\\ &\ \Rightarrow\ 2m-5\ <\ t\ominus i\ <\ 2m+1\\ &\ \Rightarrow\ t\ominus i\in\{2m-3,\ 2m-1\}\end{aligned}$$

$$\Rightarrow K_9{}' = \big\{\left(\begin{smallmatrix} a_1\,a_2\,a_3\\ b_1\,b_2\,b_3\end{smallmatrix}\right)\in\mathbb{Z}_{4m}^{3\times 2}\mid \Psi\left(\begin{smallmatrix} a_1\,a_2\,a_3\\ b_1\,b_2\,b_3\end{smallmatrix}\right)=\left(\begin{smallmatrix} a'_1\,a'_2\,a'_3\\ b'_1\,b'_2\,b'_3\end{smallmatrix}\right)\in\mathscr{K}_9,\ (x\neq y),\ a'_y=b'_y=a'_z=b'_z=1, a_y\ominus b_y=a_z\ominus b_z=0, \sum_n(a_n\ominus b_n)\in\{2m-3,2m-1\}\big\}$$

**Case 2:** Let $a'_y=b'_y=a'_z=b'_z=1$, $(x\neq y)$, $a_y\ominus b_y=0$, $a_z\ominus b_z=1$. W.l.o.g. $j\ominus k=0$, $l\ominus s=1$.

$$\begin{aligned}&\alpha\in\left]((t\ominus i)-1)\frac{\pi}{2m},\ ((t\ominus i)+1)\frac{\pi}{2m}\right[\\ &\beta\in\left[0,\ \frac{\pi}{m}\right[\\ &\gamma\in\left]((l\ominus s)-2)\frac{\pi}{2m},\ ((l\ominus s)+2)\frac{\pi}{2m}\right[\end{aligned}$$

$$\pi=\alpha+\beta+\gamma$$

$$\Rightarrow\ \pi \in \left]((t\ominus i)-1+(l\ominus s)-2)\frac{\pi}{2m},\quad ((t\ominus i)+1+2+(l\ominus s)+2)\frac{\pi}{2m}\right[$$
$$\Rightarrow\ 2m-5\ <\ (t\ominus i)+(l\ominus s)\ <\ 2m+3$$
$$\Rightarrow\ (t\ominus i)+(l\ominus s)\in\{2m-3,\ 2m-1,\ 2m+1\}$$

$$\Rightarrow K_9'' = \Big\{\left(\begin{smallmatrix}a_1\,a_2\,a_3\\ b_1\,b_2\,b_3\end{smallmatrix}\right)\in\mathbb{Z}_{4m}^{3\times2}\mid \Psi\left(\begin{smallmatrix}a_1\,a_2\,a_3\\ b_1\,b_2\,b_3\end{smallmatrix}\right)=\left(\begin{smallmatrix}a'_1\,a'_2\,a'_3\\ b'_1\,b'_2\,b'_3\end{smallmatrix}\right)\in\mathscr{K}_9,\ (x\neq y),$$
$$a'_y=b'_y=a'_z=b'_z=1, a_y\ominus b_y=a_z\ominus b_z=1,$$
$$\sum_n(a_n\ominus b_n)\in\{2m-3,2m-1,2m+1\}\Big\}$$

**Case 3:** Let $a'_y=b'_y=a'_z=b'_z=1$, $(x\neq y)$, $a_y\ominus b_y=a_z\ominus b_z\neq 0$. W.l.o.g. $j\ominus k\neq 0$, $l\ominus s\neq 0$.

$$\alpha\in\left]((t\ominus i)-1)\frac{\pi}{2m},\ ((t\ominus i)+1)\frac{\pi}{2m}\right[$$
$$\beta\in\left]((j\ominus k)-2)\frac{\pi}{2m},\ ((j\ominus k)+2)\frac{\pi}{2m}\right[$$
$$\gamma\in\left]((l\ominus s)-2)\frac{\pi}{2m},\ ((l\ominus s)+2)\frac{\pi}{2m}\right[$$

$$\pi=\alpha+\beta+\gamma$$
$$\Rightarrow\ \pi\in\left]((t\ominus i)+(j\ominus k)+(l\ominus s)-5)\frac{\pi}{2m},\quad ((t\ominus i)+(j\ominus k)+(l\ominus s)+5)\frac{\pi}{2m}\right[$$
$$\Rightarrow\ 2m-5\ <\ (t\ominus i)+(j\ominus k)+(l\ominus s)\ <\ 2m+5$$
$$\Rightarrow\ (t\ominus i)+(j\ominus k)+(l\ominus s)\in\{2m-3,\ 2m-1,\ 2m+1,\ 2m+3\}$$

$$\Rightarrow K_9''' = \Big\{\left(\begin{smallmatrix}a_1\,a_2\,a_3\\ b_1\,b_2\,b_3\end{smallmatrix}\right)\in\mathbb{Z}_{4m}^{3\times2}\mid \Psi\left(\begin{smallmatrix}a_1\,a_2\,a_3\\ b_1\,b_2\,b_3\end{smallmatrix}\right)=\left(\begin{smallmatrix}a'_1\,a'_2\,a'_3\\ b'_1\,b'_2\,b'_3\end{smallmatrix}\right)\in\mathscr{K}_9,\ (x\neq y),$$
$$a'_y=b'_y=a'_z=b'_z=1,\ a_y\ominus b_y\neq 0,\ a_z\ominus b_z\neq 0,$$
$$\sum_n(a_n\ominus b_n)\in\{2m-3,2m-1,2m+1,2m+3\}\Big\}$$

Summed up we get:

$$K_9 = K_9'\cup K_9''\cup K_9'''$$

### 3.1.10 $\mathscr{K}_{10}$:

$M=\left(\begin{smallmatrix}t\,j\,l\\ i\,k\,s\end{smallmatrix}\right)=\left(\begin{smallmatrix}a_1\,a_2\,a_3\\ b_1\,b_2\,b_3\end{smallmatrix}\right)\in\mathbb{Z}_{4m}^{3\times2}$ and $M'=\Psi(M)=\left(\begin{smallmatrix}a'_1\,a'_2\,a'_3\\ b'_1\,b'_2\,b'_3\end{smallmatrix}\right)\in\mathscr{K}_{10}$. $M'=\left(\begin{smallmatrix}1\,1\,1\\ 1\,1\,1\end{smallmatrix}\right)$. Four cases can be distinguished:

**Case 1:** The variables in each variable pair are not equal: $a_n \ominus b_n \neq 0 \; \forall n$.

$$\alpha \in \left]((t \ominus i) - 2)\frac{\pi}{2m},\ ((t \ominus i) + 2)\frac{\pi}{2m}\right[$$
$$\beta \in \left]((j \ominus k) - 2)\frac{\pi}{2m},\ ((j \ominus k) + 2)\frac{\pi}{2m}\right[$$
$$\gamma \in \left]((l \ominus s) - 2)\frac{\pi}{2m},\ ((l \ominus s) + 2)\frac{\pi}{2m}\right[$$

$$\begin{aligned}
&\pi = \alpha + \beta + \gamma \\
&\Rightarrow\ \pi \in \left]((t \ominus i) + (j \ominus k) + (l \ominus s) - 6)\frac{\pi}{2m},\quad ((t \ominus i) + (j \ominus k) + (l \ominus s) + 6)\frac{\pi}{2m}\right[ \\
&\Rightarrow\ 2m - 6\ <\ (t \ominus i) + (j \ominus k) + (l \ominus s)\ <\ 2m + 6 \\
&\Rightarrow\ (t \ominus i) + (j \ominus k) + (l \ominus s) \in \{2m-4,\ 2m-2,\ 2m,\ 2m+2,\ 2m+4\}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow K_{10}{}' = \Big\{ \tbinom{t\,j\,l}{i\,k\,s} \in \mathbb{Z}_{4m}^{3\times 2} \mid \Psi\tbinom{t\,j\,l}{i\,k\,s} \in \mathscr{K}_{10},\ t \ominus i \neq 0,\ j \ominus k \neq 0,\ l \ominus s \neq 0, \\
(t \ominus i) + (j \ominus k) + (l \ominus s) \in \{2m-4, 2m-2, 2m, 2m+2, 2m+4\} \Big\}
\end{aligned}$$

**Case 2:** The variables of exactly one variable pair are equal: $a_x \ominus b_x \neq 0$, $a_y \ominus b_y \neq 0$, $a_z \ominus b_z = 0$. W.l.o.g. $\binom{t\,j\,l}{i\,k\,s} = \binom{a_1\,a_2\,a_3}{b_1\,b_2\,b_3}$.

$$\alpha \in \left]((t \ominus i) - 2)\frac{\pi}{2m},\ ((t \ominus i) + 2)\frac{\pi}{2m}\right[$$
$$\beta \in \left]((j \ominus k) - 2)\frac{\pi}{2m},\ ((j \ominus k) + 2)\frac{\pi}{2m}\right[$$
$$\gamma \in \left[0,\ \frac{\pi}{m}\right[$$

$$\begin{aligned}
&\pi = \alpha + \beta + \gamma \\
&\Rightarrow\ \pi \in \left]((t \ominus i) - 2 + (j \ominus k) - 2)\frac{\pi}{2m},\quad ((t \ominus i) + 2 + (j \ominus k) + 2 + 2)\frac{\pi}{2m}\right[ \\
&\Rightarrow\ 2m - 6\ <\ (t \ominus i) + (j \ominus k)\ <\ 2m + 4 \\
&\Rightarrow\ (t \ominus i) + (j \ominus k) \in \{2m-4,\ 2m-2,\ 2m,\ 2m+2\}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow K_{10}{}'' = \Big\{ \tbinom{a_1\,a_2\,a_3}{b_1\,b_2\,b_3} \in \mathbb{Z}_{4m}^{3\times 2} \mid \Psi\tbinom{a_1\,a_2\,a_3}{b_1\,b_2\,b_3} = \tbinom{a_1'\,a_2'\,a_3'}{b_1'\,b_2'\,b_3'} \in \mathscr{K}_9,\ (x \neq y \neq z), \\
a_x \ominus b_x \neq 0,\ a_y \ominus b_y \neq 0,\ a_z \ominus b_z = 0, \\
\sum_n (a_n \ominus b_n) \in \{2m-4, 2m-2, 2m, 2m+2\} \Big\}
\end{aligned}$$

**Case 3:** The variables of exactly two variable pairs are equal: $a_x \ominus b_x \neq 0$, $a_y \ominus b_y = 0$, $a_z \ominus b_z = 0$. W.l.o.g. $\binom{t\,j\,l}{i\,k\,s} = \binom{a_1\,a_2\,a_3}{b_1\,b_2\,b_3}$.

$$\alpha \in \left]((t \ominus i) - 2)\frac{\pi}{2m},\ ((t \ominus i) + 2)\frac{\pi}{2m}\right[ \quad ; \quad \beta \in \left[0,\ \frac{\pi}{m}\right[ \quad ; \quad \gamma \in \left[0,\ \frac{\pi}{m}\right[$$

$$\begin{aligned}
&\pi = \alpha + \beta + \gamma \\
&\Rightarrow\ \pi \in \left]((t \ominus i) - 2)\frac{\pi}{2m},\quad ((t \ominus i) + 2 + 2 + 2)\frac{\pi}{2m}\right[ \\
&\Rightarrow\ 2m - 6\ <\ t \ominus i\ <\ 2m + 2 \\
&\Rightarrow\ t \ominus i \in \{2m-4,\ 2m-2,\ 2m\}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow K_{10}''' = \Big\{ \tbinom{a_1\,a_2\,a_3}{b_1\,b_2\,b_3} \in \mathbb{Z}_{4m}^{3\times 2} \mid \Psi\tbinom{a_1\,a_2\,a_3}{b_1\,b_2\,b_3} = \tbinom{a'_1\,a'_2\,a'_3}{b'_1\,b'_2\,b'_3} \in \mathscr{K}_9,\ (x \neq y \neq z),& \\
a_x \ominus b_x \neq 0,\ \ a_y \ominus b_y = 0,\ \ a_z \ominus b_z = 0,& \\
\sum_n (a_n \ominus b_n) \in \{2m-4, 2m-2, 2m\}\Big\}&
\end{aligned}$$

**Case 4:** The variables of each variable pair are equal: $a_n \ominus b_n = 0\ \forall n$.

$$\alpha \in \left[0,\ \frac{\pi}{m}\right[ \quad ; \quad \beta \in \left[0,\ \frac{\pi}{m}\right[ \quad ; \quad \gamma \in \left[0,\ \frac{\pi}{m}\right[$$

$$\begin{aligned}
&\pi = \alpha + \beta + \gamma \\
&\Rightarrow\ \pi \in \left[0,\ (2 + 2 + 2)\frac{\pi}{2m}\right[ \\
&\Rightarrow\ 0\ \leq\ \pi\ <\ \frac{3\pi}{m} \\
&\Rightarrow\ 0\ \leq\ m\ <\ 3
\end{aligned}$$

Obviously, this case only exists for $\mathcal{OPRA}_1$ and $\mathcal{OPRA}_2$.

$$\Rightarrow K_{10}'''' = \ \Big\{ \tbinom{t\,j\,l}{i\,k\,s} \in \mathbb{Z}_{4m}^{3\times 2} \mid \Psi\tbinom{t\,j\,l}{i\,k\,s} \in \mathscr{K}_{10},\ t \ominus i, j \ominus k, l \ominus s = 0 \Big\}$$

Altogether we get:

$$K_{10} = \begin{cases} K_{10}' \cup K_{10}'' \cup K_{10}''' & \text{, if } m \geq 3 \\ K_{10}' \cup K_{10}'' \cup K_{10}''' \cup K_{10}'''' & \text{, if } m < 3 \end{cases}$$

### 3.1.11 Negatively oriented triangles

If the triangle is negatively oriented, the corresponding set $\tilde{K_n}$ can be derived by substitution of variables. We define a transformation $\tilde{\Psi} : \mathbb{Z}_{4m}^{3\times 2} \to \mathbb{Z}_2^{3\times 2}$:

$$\tilde{\Psi}\begin{pmatrix} t & j & l \\ i & k & s \end{pmatrix} = \Psi\begin{pmatrix} i & k & s \\ t & j & l \end{pmatrix}$$

$\tilde{K_n}$ can now be derived analogously to $K_n$ just by considering $\tilde{\Psi}$ instead of $\Psi$.

## 3.2 Configurations with "same" relations: all o-points at the same position

In this section we look at the configuration in which all o-points are at the same position ($A = B = C$, see Figure 3). We assume the o-points to be positively oriented. We have the $\mathcal{OPRA}_m$ relations $\vec{A}\ {}_m\angle i\ \vec{B}$, $\vec{B}\ {}_m\angle k\ \vec{C}$, and $\vec{C}\ {}_m\angle s\ \vec{A}$.

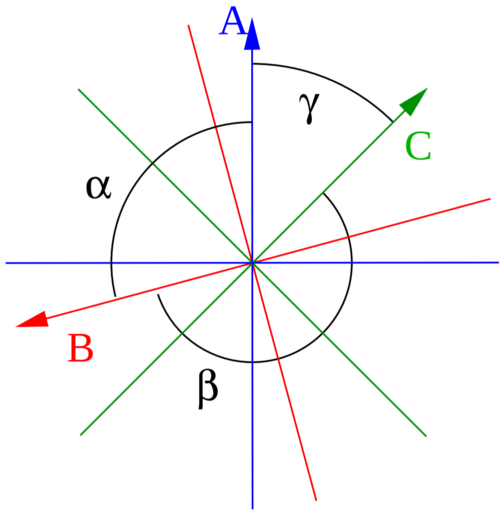


**Figure 3:** Composition with only "same" relations: All o-points are at the same position.

For the triples $(i, k, s)$ we define a transformation $\Psi' : \mathbb{Z}_{4m}^3 \to \mathbb{Z}_2^3$:

$$\Psi'(i, k, s) = (i \bmod 2,\ k \bmod 2,\ s \bmod 2)$$

We can now identify four different classes $\mathscr{S}_1, \ldots, \mathscr{S}_4$ for a matrix $A = \Psi'(i, k, s)$:

$$\mathscr{S}_1 = \{(0,0,0)\}$$
$$\mathscr{S}_2 = \{(0,0,1),\ (0,1,0),\ (1,0,0)\}$$
$$\mathscr{S}_3 = \{(0,1,1),\ (1,0,1),\ (1,1,0)\}$$

$$\mathscr{S}_4 = \{(1,1,1)\}$$

In the following, we will take a closer look at those classes and will specify the set $S_i$ of valid configurations for each one.

### 3.2.1 $\mathscr{S}_1$ :

The configuration $i = 0, k = 0, s = 0$ is trivially valid. Otherwise holds:

$$\alpha = i\frac{\pi}{2m} \quad ; \quad \beta = k\frac{\pi}{2m} \quad ; \quad \gamma = s\frac{\pi}{2m}$$

$$2\pi = \alpha + \beta + \gamma = {\scriptstyle (i+k+s)}\frac{\pi}{2m}$$
$$\Rightarrow i + k + s = 4m$$

$$\Rightarrow S_1 = \left\{(i,k,s) \in \mathbb{Z}^3_{4m} \mid \Psi'(i,k,s) \in \mathscr{S}_1,\ i+k+s = 4m \text{ or } i,k,s = 0\right\}$$

### 3.2.2 $\mathscr{S}_2$ :

W.l.o.g., $\Psi'(i,k,s) = (1,0,0)$ in $\mathbb{Z}_2$.

$$\alpha \in \left]{\scriptstyle (i-1)}\frac{\pi}{2m},\ {\scriptstyle (i+1)}\frac{\pi}{2m}\right[ \quad ; \quad \beta = {\scriptstyle k}\frac{\pi}{2m} \quad ; \quad \gamma = {\scriptstyle s}\frac{\pi}{2m}$$

$$\begin{aligned} 2\pi &= \alpha + \beta + \gamma \\ &\Rightarrow 2\pi \in \left](i-1+k+s)\frac{\pi}{2m},\ (i+1+k+s)\frac{\pi}{2m}\right[ \\ &\Rightarrow 4m-1 \ < \ i+k+s \ < \ 4m+1 \\ &\Rightarrow i+k+s = 4m, \quad \text{but } i+k+s \equiv_2 1 \quad \lightning \end{aligned}$$

$\mathscr{S}_2$ is no valid configuration class, $S_2 = \emptyset$.

### 3.2.3 $\mathscr{S}_3$ :

W.l.o.g., $\Psi'(i,k,s) = (1,1,0)$ in $\mathbb{Z}_2$.

$$\begin{aligned} \alpha &\in \left](i-1)\frac{\pi}{2m},\ (i+1)\frac{\pi}{2m}\right[ \\ \beta &\in \left](k-1)\frac{\pi}{2m},\ (k+1)\frac{\pi}{2m}\right[ \\ \gamma &= s\frac{\pi}{2m} \end{aligned}$$

$$
\begin{aligned}
2\pi =\ & \alpha + \beta + \gamma \\
\Rightarrow 2\pi \in & \left]( i - 1 + k - 1 + s)\frac{\pi}{2m},\ (i + 1 + k + 1 + s)\frac{\pi}{2m}\right[ \\
\Rightarrow\ & 4m - 2 \ < \ i + k + s \ < \ 4m + 2 \\
\Rightarrow\ & i + k + s = 4m, \text{ because } i + k + s \equiv_2 0
\end{aligned}
$$

$$\Rightarrow S_3 = \left\{(i,k,s) \in \mathbb{Z}^3_{4m} \mid \Psi'(i,k,s) \in \mathscr{S}_3\ i + k + s = 4m\right\}$$

### 3.2.4 $\mathscr{S}_4$ :

$$
\begin{aligned}
\alpha &\in \left](i-1)\frac{\pi}{2m},\ (i+1)\frac{\pi}{2m}\right[ \\
\beta &\in \left](k-1)\frac{\pi}{2m},\ (k+1)\frac{\pi}{2m}\right[ \\
\gamma &\in \left](s-1)\frac{\pi}{2m},\ (s+1)\frac{\pi}{2m}\right[
\end{aligned}
$$

$$
\begin{aligned}
2\pi =\ & \alpha + \beta + \gamma \\
\Rightarrow\ 2\pi \in & \left](i - 1 + k - 1 + s - 1)\frac{\pi}{2m},\ (i + 1 + k + 1 + s + 1)\frac{\pi}{2m}\right[ \\
\Rightarrow\ & 4m - 3 \ < \ i + k + s \ < \ 4m + 3 \\
\Rightarrow\ & i + k + s \in \{4m - 1,\ 4m + 1\}, \text{ because } i + k + s \equiv_2 1
\end{aligned}
$$

$$\Rightarrow S_4 = \left\{(i,k,s) \in \mathbb{Z}^3_{4m} \mid \Psi'(i,k,s) \in \mathscr{S}_4,\ i + k + s \in \{4m - 1,\ 4m + 1\}\right\}$$

### 3.2.5 Negatively oriented o-points

In the case of negatively oriented o-points, we again use substitution. We define a transformation $\tilde{\Psi}' : \mathbb{Z}^3_{4m} \to \mathbb{Z}^3_2$:

$$\tilde{\Psi}'(i,k,s) = \Psi'(4m - i, 4m - i, 4m - s)$$

and derive $\tilde{S}_n$ analogously to $S_n$ by considering $\tilde{\Psi}'$ instead of $\Psi'$.

## 3.3 Configurations with "same" relations: two of three o-points at the same position

W.l.o.g, $A$ and $B$ are at the same position, and $C$ is not (see Figure 4). We have the configuration $A\ m\angle i\ B$, $B\ m\angle^l_k\ C$, $C\ m\angle^t_s\ A$. For a valid configuration, $l$ must equal $s$, and because of their lacking influence on the configuration, their values are arbitrary.

Regarding $i$, $k$ and $t$, we can identify four classes $\mathscr{T}_1, \ldots, \mathscr{T}_4$ for $A = \Psi'(i,k,t)$:

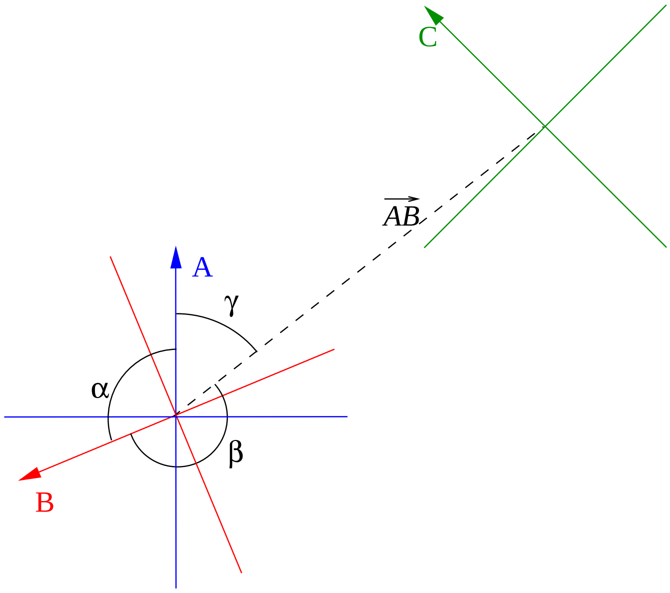


**Figure 4:** Composition with one "same" relation: $\vec{A}$ and $\vec{B}$ share the same position, $\vec{C}$ does not.

$$\mathscr{T}_1 = \{(0,0,0)\}$$
$$\mathscr{T}_2 = \{(0,0,1),\ (0,1,0),\ (1,0,0)\}$$
$$\mathscr{T}_3 = \{(0,1,1),\ (1,0,1),\ (1,1,0)\}$$
$$\mathscr{T}_4 = \{(1,1,1)\}$$

These are the same sets as $\mathscr{S}_1, \ldots, \mathscr{S}_4$, but note that we consider $t$ and not $s$ to determine the classes in this case.

In the following, we will specify a set $T_i$ of valid configurations for each of those configurations. Again, we assume the configuration to be positively oriented.

### 3.3.1 $\mathscr{T}_1$ :

**Case 1:** Let $t = 0$. The configuration $i = k = t = 0$ is trivially valid. For $i \neq 0$ or $k \neq 0$ holds:

$$\alpha = i\frac{\pi}{2m} \quad ; \quad \beta = k\frac{\pi}{2m} \quad ; \quad \gamma = 0$$

$$2\pi = \alpha + \beta + \gamma = (i+k)\frac{\pi}{2m}$$
$$\Rightarrow i + k = 4m$$

$$\Rightarrow\ T_1' = \left\{(i,k,t) \in \mathbb{Z}_{4m}^3 \mid \Psi'(i,k,t) \in \mathscr{T}_1, i+k=4m\right\}$$

**Case 2:** Let $t \neq 0$. It holds:

$$\alpha = i\frac{\pi}{2m} \quad ; \quad \beta = k\frac{\pi}{2m} \quad ; \quad \gamma = (4m-t)\frac{\pi}{2m}$$

$$2\pi = \alpha+\beta+\gamma = (i+k+4m-t)\frac{\pi}{2m}$$
$$\Rightarrow i+k-t=0$$

$$\Rightarrow T_1'' = \left\{(i,k,t) \in \mathbb{Z}_{4m}^3 \mid \Psi'(i,k,t) \in \mathscr{T}_1,\ i+k-t=0\right\}$$

Altogether we get:

$$T_1 = \left\{(i,k,t) \in \mathbb{Z}_{4m}^3 \mid \Psi'(i,k,t) \in \mathscr{T}_1,\ i,k,t=0 \text{ or } i+k=4m \text{ or } i+k-t=0\right\}$$

### 3.3.2 $\mathscr{T}_2$ :

W.l.o.g., $\Psi'(i,k,t) = (1,0,0)$.

$$\alpha \in \left](i-1)\frac{\pi}{2m},\ (i+1)\frac{\pi}{2m}\right[ \quad ; \quad \beta = k\frac{\pi}{2m} \quad ; \quad \gamma = (4m-t)\frac{\pi}{2m}$$

$$2\pi = \alpha+\beta+\gamma \in \left](i-1+k+4m-t)\frac{\pi}{2m},\ (i+1+k+4m-t)\frac{\pi}{2m}\right[$$
$$\Rightarrow\ -1\ <\ i+k-t\ <\ 1$$
$$\Rightarrow i+k-t=0, \quad \text{but } i+k+t \equiv_2 1 \quad ⚡$$

$\mathscr{T}_2$ is no valid configuration class, $T_2 = \emptyset$.

### 3.3.3 $\mathscr{T}_3$ :

W.l.o.g., $\Psi'(i,k,t) = (1,1,0)$.

**Case 1: Let $t = 0$.**

$$\alpha \in \left](i-1)\frac{\pi}{2m},\ (i+1)\frac{\pi}{2m}\right[ \quad ; \quad \beta \in \left](k-1)\frac{\pi}{2m},\ (k+1)\frac{\pi}{2m}\right[ \quad ; \quad \gamma = 0$$

$$2\pi = \alpha+\beta+\gamma \in \left](i-1+k-1)\frac{\pi}{2m},\ (i+1+k+1)\frac{\pi}{2m}\right[$$
$$\Rightarrow 4m-2\ <\ i+k\ <\ 4m+2$$
$$\Rightarrow i+k=4m, \quad \text{because } i+k \equiv_2 0$$

$$\Rightarrow\ T_3' = \left\{(i,k,t) \in \mathbb{Z}_{4m}^3 \mid \Psi'(i,k,t) \in \mathscr{T}_3,\ i+k=4m\right\}$$

**Case 2: Let $t \neq 0$.**

$$\begin{aligned} \alpha &\in \left](i-1)\frac{\pi}{2m},\ (i+1)\frac{\pi}{2m}\right[ \\ \beta &\in \left](k-1)\frac{\pi}{2m},\ (k+1)\frac{\pi}{2m}\right[ \\ \gamma &= (4m-t)\frac{\pi}{2m} \end{aligned}$$

$$\begin{aligned} 2\pi = \ & \alpha+\beta+\gamma \in \left](i-1+k-1+4m-t)\frac{\pi}{2m},\ (i+1+k+1+4m-t)\frac{\pi}{2m}\right[ \\ \Rightarrow\ & -2 \ < \ i+k-t \ < \ 2 \\ \Rightarrow\ & i+k-t=0, \quad \text{because } i+k+t \equiv_2 0 \end{aligned}$$

$$\Rightarrow\ T_3'' = \left\{(i,k,t) \in \mathbb{Z}_{4m}^3 \mid \Psi'(i,k,t) \in \mathscr{T}_3,\ i+k-t=0\right\}$$

Altogether we get:

$$T_3 = \left\{(i,k,t) \in \mathbb{Z}_{4m}^3 \mid \Psi'(i,k,t) \in \mathscr{T}_3,\ i+k=4m \text{ or } i+k-t=0\right\}$$

### 3.3.4 $\mathscr{T}_4$ :

$$\begin{aligned} \alpha &\in \left](i-1)\frac{\pi}{2m},\ (i+1)\frac{\pi}{2m}\right[ \\ \beta &\in \left](k-1)\frac{\pi}{2m},\ (k+1)\frac{\pi}{2m}\right[ \\ \gamma &\in \left](4m-t-1)\frac{\pi}{2m},\ (4m-t+1)\frac{\pi}{2m}\right[ \end{aligned}$$

$$\begin{aligned} 2\pi = \ & \alpha+\beta+\gamma \in \left](i+k+4m-t-3)\frac{\pi}{2m},\ (i+k+4m-t+3)\frac{\pi}{2m}\right[ \\ \Rightarrow\ & -3 \ < \ i+k-t \ < \ 3 \\ \Rightarrow\ & i+k-t \in \{-1,\ 1\}, \quad \text{because } i+k+t \equiv_2 1 \end{aligned}$$

$$\Rightarrow\ T_4 = \left\{(i,k,t) \in \mathbb{Z}_{4m}^3 \mid \Psi'(i,k,t) \in \mathscr{T}_4,\ i+k-t \in \{-1,\ 1\}\right\}$$

### 3.3.5 Negatively oriented o-points

In the case of negatively oriented o-points, we derive $\tilde{T}_n$ by substitution analogously to section 3.2.5.

**Algorithm 1** The $\mathcal{OPRA}_m$ composition algorithm

```
R = ∅
if r1 = m∠i ∧ r2 = m∠k then                                ▷ only "same" relations
    calculate R according to (4)
else
    if r1 = m∠i ∧ r2 = m∠_k^l then                          ▷ one "same" relation
        calculate R according to (5)
    else
        if r1 = m∠_i^j ∧ r2 = m∠k then                      ▷ one "same" relation
            calculate R according to (6)
        else                                  ▷ usual case – no "same" relations
            for t = 0 to 4m − 1 do
                begin
                for s = 0 to 4m − 1 do                   ▷ check "non-same" results
                    begin
                    if (t j l / i k s) ∈ K_n then
                        R = R ∪ m∠_t^s
                    end
                if (i, k, t) ∈ T_n' then                      ▷ check "same" results
                    R = R ∪ m∠t
                end
return R
```

## 3.4 Calculating the composition

After having identified the valid configuration classes $\mathscr{K}_n$, $\mathscr{S}_n$, and $\mathscr{T}_n$, calculating the result of the composition operation is straight-forward. Given two $\mathcal{OPRA}_m$ relations $r_1 = {}_m\angle_i^j$ and $r_2 = {}_m\angle_k^l$, the relation ${}_m\angle_s^t$ is part of the solution of $r_1 \circ r_2$, if $\binom{t\,j\,l}{i\,k\,s}$ is a member of one of the sets $K_1, \ldots, K_{10}$ or $(i, k, t)$ is a member of $T_1, \ldots, T_4$, respectively. Thus, we need to regard $\binom{t\,j\,l}{i\,k\,s}$ and $(i, k, t)$ for all $4m$ possible values of $s$ and $t$, determine the corresponding classes $\mathscr{K}_n$ and $\mathscr{T}_n$, and regard the resulting sets $K_n$ and $T_n$. For compositions including one or two "same" relations, we can act accordingly and determine $T_n$ or $S_n$. However, as those compositions reduce to simple addition of angular intervals, they can be computed easily in $O(1)$ regardless of $m$. The resulting composition results are

$$
{}_m\angle i \circ {}_m\angle k = \begin{cases} {}_m\angle(i+k) & i \equiv_2 0 \vee k \equiv_2 0 \\ \bigcup_{a=i+k-1}^{i+k+1} {}_m\angle a & \text{else} \end{cases}, \tag{4}
$$

$$ {}_m\angle i \circ {}_m\angle_k^l = \begin{cases} {}_m\angle_{i+k}^l & i \equiv_2 0 \vee k \equiv_2 0 \\ \bigcup_{a=i+k-1}^{i+k+1} {}_m\angle_a^l & \text{else} \end{cases} , \tag{5} $$

$$ {}_m\angle_i^j \circ {}_m\angle k = \begin{cases} {}_m\angle_i^{j-k} & j \equiv_2 0 \vee k \equiv_2 0 \\ \bigcup_{a=j-k-1}^{j-k+1} {}_m\angle_i^a & \text{else} \end{cases} \tag{6} $$

The complete algorithm (for positively oriented o-points) is given on page 24. If the o-points are not positively oriented, use substitution before as described in Sections 3.1.11, 3.2.5, and 3.3.5.

In the worst case of $r_1$ and $r_2$ being no "same" relations, the algorithm loops $4m \cdot (4m + 1)$ times. Calculation of $K_n$, $S_n$, and $T_n$ is done in constant time for each configuration, so the overall complexity of the algorithm is $O(m^2)$. Because $m$ is constact, the algorithm computes composition for every instance of $\mathcal{OPRA}_m$ in constant time.

A reference implementation of this algorithm is part of the SparQ toolbox[3] (Wallgrün et al., 2006).

## 4 Algebraic closure and closure under constraints

The $\mathcal{OPRA}_m$ composition algorithm as described in the previous section has been included into the SparQ toolbox (Wallgrün et al., 2006). Unfortunately, the standard constraint reasoning procedures algebraic closure and backtracking search over atomic networks based on algebraic closure can only be used for approximate consistency checking. The reason is that for $\mathcal{OPRA}_m$ algebraic closure is not sufficient to decide consistency even for atomic networks and thus inconsistencies may not be discovered.

That algebraic closure is insufficient to decide consistency even for atomic networks can be shown using the *closure under constraints* criterion recently introduced by Renz and Ligozat (2005). We quote the two important definitions from their paper (Definition 1 and 2, Section 3):

> **Refinement to a subatomic relation:** Let $\Theta$ be a consistent atomic CSP over a set $\mathcal{A}$ and $xRy \in \Theta$ a constraint. Let $R'$ be the union of all tuples $(u, v) \in R$ that can be instantiated to $x$ and $y$ as part of a solution of $\Theta$. If $R' \subset R$, then $\Theta$ refines $R$ to the subatomic relation $R'$.
>
> **Closure under constraints:** Let $\mathcal{A}$ be a set of atomic relations. $\mathcal{A}$ is closed under constraints if no relation $R \in \mathcal{A}$ can be refined to non-overlapping subatomic relation, i.e., if for each $R \in \mathcal{A}$ all subatomic relations $R' \subset R$ to which $R$ can be refined have a nonempty intersection.

Renz and Ligozat show that algebraic closure decides consistency for CSPs over $\mathcal{A}$ if and only if $\mathcal{A}$ is closed under constraints (Theorem 1, Section 3).

[3] SparQ can be obtained at `http://www.sfbtr8.uni-bremen.de/project/r3/sparq/`

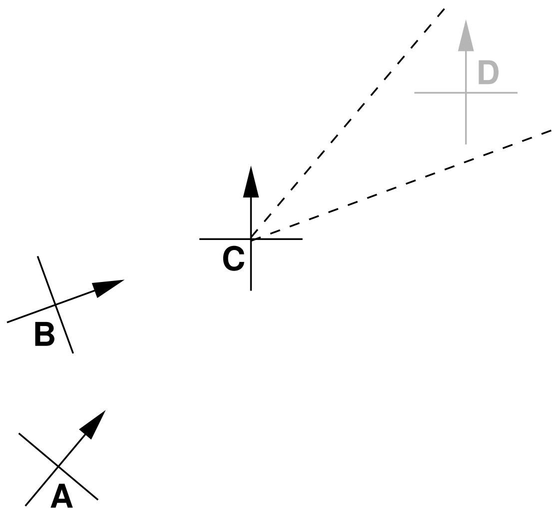


**Figure 5:** $\mathcal{OPRA}_m$ is not closed under constraints as the relation between $\vec{C}$ and $\vec{D}$ can be refined to two subatomic relations with an empty intersection (dashed lines).

If we now look at Figure 5, we see that this is not the case for $\mathcal{OPRA}_m$. The figure shows three oriented points $\vec{A}$, $\vec{B}$, and $\vec{C}$ which form a clockwise oriented triangle with $\vec{A}$ and $\vec{B}$ both pointing at $\vec{C}$. The complete triangle is described by the following $\mathcal{OPRA}_2$ relations: $\vec{A}\ {}_2\angle_1^5\ \vec{B}$, $\vec{A}\ {}_2\angle_0^3\ \vec{C}$, and $\vec{B}\ {}_2\angle_0^3\ \vec{C}$. We now add another oriented point $\vec{D}$ with $\vec{C}\ {}_2\angle_7^3\ \vec{D}$. Then we refine this relation between $\vec{C}$ and $\vec{D}$ by stating that $\vec{A}$ and $\vec{B}$ also point at $\vec{D}$: $\vec{A}\ {}_2\angle_0^3\ \vec{D}$ and $\vec{B}\ {}_2\angle_0^3\ \vec{D}$. As indicated by the dashed lines, this refines the $\vec{C}\ {}_2\angle_3^7\ \vec{D}$ relation into two different linear subatomic relations which have an empty intersection and $\vec{D}$ would have to lie on both lines at the same time which is not possible. This inconsistency is not discovered by the algebraic closure algorithm and hence this counter example demonstrates that even for atomic networks algebraic closure does not decide consistency for $\mathcal{OPRA}_m$.

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## References

Reinhard Moratz. Qualitative spatial reasoning about oriented points. Technical Report 003/10-2004, SFB/TR 8 Spatial Cognition, October 2004.

Reinhard Moratz. Representing relative direction as binary relation of oriented points. In *Proceedings of the 17th European Conference on Artificial Intelligence (ECAI 2006)*, Riva del Garda, Italy, August 2006.

Reinhard Moratz, Frank Dylla, and Lutz Frommberger. A relative orientation algebra with adjustable granularity. In *Proceedings of the Workshop on Agents in Real-Time and Dynamic Environments (IJCAI 05)*, Edinburgh, Scotland, July 2005.

Jochen Renz and Gérard Ligozat. Weak composition for qualitative spatial and temporal reasoning. In *Proceedings of the 11th International Conference on Principles and Practice of Constraint Programming (CP 2005)*, pages 534–548, Sitges (Barcelona), Spain, October 2005.

Jan Oliver Wallgrün, Lutz Frommberger, Diedrich Wolter, Frank Dylla, and Christian Freksa. A toolbox for qualitative spatial representation and reasoning. In *Spatial Cognition V: Reasoning, Action, Interaction: International Conference Spatial Cognition*, Bremen, Germany, 2006.