# **Towards Spatial Reasoning with Strings and Pins**

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## Abstract

Spatial problems and a variety of abstract problems can be solved by humans with more ease if the problem can be visualized and/or manipulated. Elaborating on the differences and interplay between representing spatial problems and solving them, this paper focuses on exploring the qualities of a strings and pins problem solving domain. This domain is compared to the straightedge and compass domain of Euclidean geometry. The strings and pins domain is described and two case studies are analyzed: (i) a strings and pins solution of the shortest path problem and (ii) a strings and pins solution to the Voronoi diagram problem; both approaches are compared to a formal solution. Differences and similarities between the straightedge and compass domain are analyzed, and features and limitations of constructive and depictive geometry are discussed. Different problem solving phases are distinguished. Furthermore, cognitive aspects of different problem solving approaches are discussed.

# 1. Introduction: Understanding through seeing, doing, and abstract thinking

It is well known that most people 'grasp' spatial problems and their solutions more easily if they can visualize them or carry out physical actions and observe their effects than purely on the basis of textual or formal descriptions (e.g., Kozhevnikov et al. 2007). This phenomenon is not restricted to spatial problems; trained theoreticians advise their students to first try to understand abstract problems through concrete diagrams, i.e. spatializing them, before tackling them formally (Polya 1945, Kaiser 2005), even though the depiction may be much more constrained than the abstract problem to be solved.

This is an indication that spatial cognition including perception and action may support insight into problems in certain situations more effectively than linear (textual or formal) descriptions, which in turn are easier to handle analytically or by computer programs. Therefore, to understand the potentials of cognitive systems for grasping and solving problems, it appears worthwhile to investigate spatial challenges and capabilities of cognitive systems to a much larger extent (Freksa 2015).

The connection between formal analysis and spatial construction probably is most widely known through the relation between formal and depictive<sup>1</sup> geometry (Euclid 1956, Larkin and Simon 1987). Although proofs in depictive geometry can be formally described, formal proofs generally are structurally not fully equivalent to the corresponding depictive proofs. The reason is that the spatial medium of a diagram (typically a piece of paper or a board) *implicitly* contributes relevant spatial relationships and dependencies that must be *explicitly* described in the formal approach, as spatial relations between formulae on the paper are not a part of the formal representation. Thus, although both the depictive and the formal representation may be given on the same medium, in the depictive case the structure of the medium is a relevant integral part of the representation, while in the formal case it is not.

It is also likely that some proofs themselves have been found with the help of insights coming from visuospatial understanding, and that they could not have been found as easily on a purely formal basis. This is in line with the long history of visual thinking in mathematics; see for example (Giaquinto, 2007). University students are still taught to visualize problems in order to gain insight for deriving general facts, which then can be formalized.

### Example 1: Geometric reasoning with diagrams vs. formal reasoning

A nice example demonstrating the difference between geometric reasoning and formal reasoning is presented by Bernays (1976), who compared Euclid's geometric approach with Hilbert's abstract theory of geometry (Mumma, 2012). Whereas in Euclid's approach an interpretation of geometric objects is necessary to understand a proof, in Hilbert's case proofs only depend on the logic form, i.e. a geometric interpretation, although possible, is not presumed. Mumma also points out that while Euclid's approach is constructive, Hilbert's formalization is existential.

One example stated in Mumma (2012) is the concept of 'opposite' that expresses that two points are located on opposite sides of a given line. Simplified, in Euler's view, i.e., by construction, two points are on opposite sides of a line if you draw them as such (Fig. 1 left). In Hilbert's case this is not sufficient. Hilbert requires the construction of an auxiliary line (l) and the existence of some point (x), which needs to be located on the line segment between  $p_1p_2$  and on (l). Formally this is given by  $OppSide(p_1, p_2, l) \leftrightarrow \exists point x B(p_1 x p_2) \land x lies on l$ , with the ternary relation B denoting that x is positioned between  $p_1$  and  $p_2$  (Fig. 1 right).



*Figure 1*. Euclid's understanding of 'opposite' (left) and Hilbert's abstract definition by means of an auxiliary line and an intersection point (right)

<sup>&</sup>lt;sup>1</sup> For the classical diagrammatic approach to Euclidean geometry, we use the term *depictive geometry* instead of *descriptive geometry* in order to avoid confusion with other (including formal) ways of describing geometric relations.

The use of diagrams in proofs by Euclid has been investigated by Manders (2008) explicating that two parts exist: the text and the diagram. Furthermore, he distinguishes exact and co-exact properties. Considering inaccuracies in manually drawn diagrams, e.g., an imperfectly drawn straight line, co-exact features are those which remain stable in the presence of these inaccuracies. Exact features are those which change in the presence of smallest variations. This is related to differences in quantitative and qualitative reasoning. By means of qualitative descriptions one can focus on distinctions between objects that make an important and relevant difference with respect to a given task (Kuipers, 1994). Reasoning is performed on a symbolic level, where small quantitative deviations may not make a difference. In comparison, in quantitative reasoning small changes of the numbers will directly influence the result.

### Example 2: Comparison between descriptive and depictive representations:

This example contrasts a 2-column table showing numbers and their squares with a graphical depiction of those numbers relating the magnitudes of these numbers spatially and visualizing them graphically. In order to understand the table (Fig. 2 left), we must mentally represent the numbers and relate their values to one another abstractly or by mental imagery. In the graph (Fig. 2 right), the values are associated through spatial relations; we can visually read off relations between the values by comparing distances; we also can directly sense slopes corresponding to relative differences of numbers in the two columns on the left. Although the concrete spatial distance is not identical to the relation between the abstract numbers, we seem to understand the values and their relations more easily through spatial reification than through number-theoretic arguments.



*Figure 2.* Descriptive (left) and depictive (right) representation of abstract information. In the descriptive representation, spatial relations in the table only relate x-values to y-values, while in the depictive representation magnitudes and variations are also spatially related and visually accessible.

It is well known that different kinds of representations afford different kinds of reasoning (Sloman 1985). In this paper, we demonstrate the particular role of spatial structures and representations that exhibit a strong correspondence with spatial structures of perception (Palmer 1999), spatial mental models (Byrne & Johnson-Laird 1989), and mental imagery (Finke 1989;

cf. also Freksa 1997). This is why spatial configurations in maps, diagrams, and in the physical environment are particularly accessible to natural and artificial cognitive systems. The purpose of our studies therefore is two-fold: (1) we investigate the potential for systematic exploitation of structure-implicit properties for problem solving and (2) we investigate how cognitive systems capable of dealing both with graphic depictions and with formal descriptions combine these abilities in such a way that they solve problems in an effective and efficient manner.

# 2. Spatial media for spatial problem solving and the strings and pins domain

Great geometric discoveries and proofs have been made for 2500 years with straightedge and compass constructions. Stimulated by the shortest path finding procedure of pulling apart the starting and the destination nodes of a street network represented by an undirected graph constructed out of strings (Dreyfus and Haugeland, 1974, Dewdney 1988, Freksa 2015 – also see section 5), the Cognitive Systems group at the Bremen Spatial Cognition Center wanted to find out whether strings can be used to represent and solve a larger interesting class of spatial problems, similarly to how straightedge and compass have been used to represent and solve an important class of problems in plane geometry.

Strings are cognitively appealing for spatial problem solving for at least two reasons: (1) they can be deformed in useful ways in which we cannot deform the lines in Euclidean geometry (e.g., Cabalar and Santos 2011); and (2) they have a length, a relevant property for distance reasoning, but this length is analytically accessible only in spatial configurations in which they exhibit a special shape. Thus we may extend the domain of spatial problem solving into problem domains for which we do not yet have a suitable theory and we may demonstrate interesting ways of distributing task components in embodied and situated cognitive systems (Freksa 2015).

In plane geometry, a straightedge represents the concept of an (infinite) straight line and a compass represents a fixed distance between a center point and an (infinite) set of points. Paper and pencil usually are not mentioned as tools in these constructions, although they are assumed to be included as reference structure and for visualization of the geometric constructions. With the straightedge and compass we can systematically construct geometric figures related to polygons and circles; we also can do qualitative spatial reasoning using straightness of Euclidean geometry, equality of distance as determined by the compass, and derived qualities such as equality and ratios of distances and angles, as well as parallelism of lines. It is important to note that the straightedge and compass that are used together with a pencil to manifest geometric constructions on sheets of paper are mere visualizations (or materializations) of the underlying conceptual structures. So a line, for instance, is considered as a one-dimensional entity without width, whereas the straight line drawn on paper must have a certain areal extension in order to be perceivable to an observer.

The same distinction between theoretical conception and physical realization applies to the domain of strings and pins presented in the following. In our conception, strings represent deformable non-elastic linear geometric objects of arbitrary shape. A deformed string maintains constant length. The length of a string can be determined by pulling the string straight. The length is then identical to the distance between the two ends of the string. Positions can be marked on strings by pins; for example, a section of a string of arbitrary length can be marked by two pins to form a *string segment*. When a string segment is pulled straight it corresponds to a line segment

in depictive geometry, whose length is identical to the distance between the pins that mark the ends of the line segment. Pins can fix string positions to a spatial medium called a *board* (similar to pencil marks on a piece of paper) or they can connect strings at given positions by collocating pins at these positions. Thus, positions can be identified either on the board or on strings, or both.

The "strings and pins" domain consists of a set of strings and pins, as well as a board. With these objects, the following operations can be performed:

- Strings can be straightened by pulling their ends apart;
- Pins can be fixed on strings;
- Pins can be fixed on the board (with or without strings attached); and
- Pins can be connected to each other.

In the following, we will use capital letters as labels to refer to strings and pins. Based on these operations, a string segment is constructed by fixing two pins A and B at two positions of a string, yielding the string segment AB. This string segment can be straightened by pulling A and B apart.

With these characteristics we can build up any geometric construction that we can draw with the straightedge and compass: a straight line (segment) is constructed by fixing a pulled string (segment) to the spatial medium with pins at both ends. A circle is constructed by fixing a string to the position that marks the center of the circle; the positions on the spatial medium where the 'loose end' of the same pulled string segment can be placed are the locations of equal distance to the center point and thus correspond to the circle line in a compass construction.

In the following subsections, two examples of how to use the strings and pins toolkit to solve geometrical problems are given. Section 2.1 presents the steps to construct a perpendicular bisector of a string segment. Section 2.2 describes the steps to build a Voronoi tessellation from three points, and how this tessellation (graph and cell areas) is rebuilt when a new point is introduced.

## 2.1 Constructing a perpendicular bisector of a string segment

We can construct the perpendicular bisector of a string segment as follows (see Fig. 3):

- Start with a pulled straight string segment AB fixed to the board (at A and B, Fig. 3a)
- Produce two pairs of equal-length string segments RS, R'S', and TU, T'U', with lengths of approximately two thirds of the length of AB (Fig. 3b)
- Fix R and T to A, and R' and T' to B (Fig. 3c)
- Connect S and S' as well as U and U' to yield S'' and U'', respectively (Fig. 3d)
- Pull straight RS'' and R'S'' and fix S'' to the board (Fig. 3e)
- Pull straight TU" and T'U" towards the opposite side of AB with respect to S"
- Fix U" to the board (Fig. 3f)
- Produce a pulled straight string segment S''U''
- S''U'' is perpendicular to AB and bisects it (Fig. 3g)

The above procedure makes use of two subroutines, which can be described as follows:

- a. Construct a straight string segment between two pins fixed to the board:
  - Start with two pins A and B fixed to the board (at different locations)
  - Take a string and fix a pin C at it
  - Connect C with A
  - Pull the string straight and fix a pin D such that D is connected to B

b. Construct string segments of equal length:

- Start with a straight string segment AB fixed to the board
- Construct a straight string segment CD between A and B
- AB and CD are of equal length



*Figure 3.* Constructing the perpendicular bisector of a string segment. Pins that are fixed to the board are marked with a black dot.

### 2.2 Voronoi diagram / tessellation obtained by the strings and pins approach

A Voronoi diagram (see Fig. 4) is a subdivision of the plane into a number of cells such that each cell contains only one point/site. The boundary of each cell is made of segments which are equidistant to the two nearest sites/points. All the Voronoi cells are convex and Voronoi vertices (nodes) are the points equidistant to three (or more) sites.

Generally, the input of a Voronoi diagram computation is a set of points (sites) and the output is a partitioning of the Euclidean plane into regions (graph and cell areas) of equal nearest neighbors. As an example, we draw the Voronoi tessellation with the strings and pins approach corresponding to the points in Fig. 5a, following the next steps:

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*Figure 4*. A Voronoi diagram<sup>2</sup>. Each site (black dot) is embedded in a colored region whose boundaries are equidistant to the closest sites.

- Step 1: Construct the perpendicular bisector of line segment AB and mark it with a string: bs(AB).
- Step 2: Construct the perpendicular bisector of line segment BC and mark it with a string: bs(BC).
- Step 3: Construct the perpendicular bisector of line segment AC and mark it with a string: bs(AC). Figure 5b shows steps 1-3.
- Step 4: Mark the intersection of all bisector strings with a pin; this is the first Voronoi vertex: V1. Note that V1 is the center of the circle that includes the 3 points A, B, and C in its circumference. Figure 5c depicts the Voronoi vertex V1.
- Step 5: Cut string bisectors from the Voronoi vertex V1.
  - How can we decide which part of a bisector remains a part of the Voronoi diagram?
  - Take string bisector (e.g., bs(AB)) and use a pin to mark a point on either side of V1. If this point is equidistant to A and B and also closer to A and B than to C, then keep this part of the string for the Voronoi diagram. If this point is closer to C than to A or B, then cut this string.
    Repeat this for the other bisectors bs(BC) and bs(AC).
- Step 6: The remaining strings mark the regions of the final Voronoi tessellation (Fig. 5d).



*Figure 5.* a) Example of three points A, B, and C in a plane; b) visualization of steps 1-3 of the Voronoi tessellation construction; c) visualization of steps 4 and 5; d) visualization of step 6: final Voronoi diagram and vertex.

<sup>&</sup>lt;sup>2</sup> Image under Creative Commons CC0 1.0 License, Universal Public Domain

Note that the procedure to obtain the perpendicular bisector using the strings and pins approach has been described in Section 2.1.

Using the strings and pins approach, we also can update a Voronoi tessellation after a new point D has been added to the problem, by following the next steps:

- Step 1: Construct the perpendicular bisector of the line segment between the new point D and the closest point, which is the one inside the same site (i.e., P=B): bs(D,P).
- Step 2: Construct the perpendicular bisector of the line segment between the new point D and all the points in the sites whose boundary has been intersected by the new obtained bisector. For example, if the bs(D,P) only intersected one boundary of a site, then only one new bisector is needed, that is, the perpendicular bisector between the new point D and the point P', neighbor of D, whose boundary has been intersected: bs(D,P'). An example of this calculation is provided in Figure 6.



*Figure 6.* Example of Voronoi tessellation obtained when a point is added to a site: a) Original Voronoi tessellation; b) new point D is added to a site; c) bs(D,P) calculation; d) bs(D,P') calculation; e) final Voronoi tessellation including a new cell with its corresponding boundary and area.

Note that the perpendicular bisector construction can be propagated to other neighbors in the following way: if the bs(D,P) intersected 2 boundaries of a site shared with neighbors P' and P" (Fig. 7a), then we need to find bs(D,P') and bs(D,P'') (Fig. 7b). If after finding bs(D,P'') this intersected a boundary of another neighbor of P", Q, then we need to find bs(D,Q), and so on, until no new boundary is intersected anymore. The consecutive steps are shown in Fig. 7c and d. Figure 7e shows the final Voronoi tessellation.



*Figure 7*. Example of the steps needed to build a Voronoi tessellation when the sites that are not neighbors of the original modified site are also affected.

# 3. Straightedge and compass vs. strings and pins

Although we can represent all straightedge and compass constructions by strings and pins, there are a few differences between depictive geometry and the strings and pins domain, relating to the perceptual interpretations of the constructions. Whereas in classical depictive geometry we record construction moves by pencil marks on paper, the strings and pins approach records line segments by placing string segments. While pencil marks are statically fixed on paper, the string segments maintain certain degrees of mobility. For example, one end of the string segment can be fixed to the board while other positions may be floating – possibly connected by pins to other floating strings. Floating strings and floating pins can be viewed as variables: their locations represent all positions they can reach within the range of their deformability.

For circle lines, the strings and pins approach does not have a static 'memory' or visualization aid; circle lines are represented by all possible locations of the loose end of a pulled string, whose other end is fixed at the center point; the length of the string corresponds to the radius of the circle. Positions marked by intersections of pencil marks in classical depictive geometry are explicitly marked by pins in the strings and pins approach.

Our pins have a special collocation feature: just as pencil marks in depictive geometry are conceived of as lines without thickness (but require thickness to be visible) several pins A, B, C, ... can be placed at identical locations on the board. In our physical model of the strings and pins domain, this is done by placing one pin on top of another. Essentially, however, strings and pins constructions are given solely by strings and pins on the flat board; i.e., conceptually, 'stacking' one pin on top of another pin does not result in a third dimension in the representation (just as two pencil lines on top of each other do not add a third dimension to a straightedge and compass construction).

But with strings and pins we can go beyond the expressivity of straightedge and compass: as strings are thought of as deformable, one end of a string segment being fixed at a given position can describe not only the outer circle line of a circle but the entire circle area, as the loose end of the radial string can reach any location inside the circle, in an unpulled configuration. We also allow that pins are fixed to the board without a string attached; in this way, pins can be used to constrain movements of strings due to their deformability in a well-defined manner (Fig. 8). Conversely, we can use constraints of deformability of strings to constrain the locations of a pin along a string, as we can slide pins along strings to describe ellipses, for example. For this, we may have to construct a string loop by joining the ends of a string segment with a floating pin.



Figure 8. Describing ellipses with strings constrained by pins.

# 4. Features and limitations of constructive and depictive geometry

A main difference between diagrammatic or physical problem solving approaches and abstract (e.g., algebraic) approaches is the specificity of the representation. For example, in diagrams, we instantiate specific triangles with sides of specific length, specific angles between them, a specific orientation, etc., even if we intend to represent a 'general triangle' that has the properties that all triangles share. This specificity is perfectly okay if we have to solve problems that concern specific entities; if we intend to make general inferences, we must argue why the special case does not restrict the general validity (Forbus et al. 2011). Polya (1945) advises on getting over the possible "wrong instantiation" fallacy by drawing and trying out a multiplicity of such examples.

From a cognitive perspective, specific and concrete instances make concepts more easily graspable (e.g., Wason 1968, Kahneman 2011); generalization follows later, possibly after exposure to a variety of specific instances (Aamodt and Plaza, 1994).

In abstract representations such as algebra, we can leave parameters variable in order to represent classes of instances. In fact, the algebraic language is so general that we must take special measures to make sure that what we describe is limited to triangles, by putting explicit constraints on the parameters. In other words, formal representations essentially require that the problem to be solved already is largely understood; otherwise, a correct formalization would be very difficult.

Thus, in diagrammatic or physical representations we must be concerned about the generality of the inferences we may make while in formal approaches we are concerned about their specificity. This demonstrates the complementarity of depictive and descriptive representations and the potential power of combining them: whereas depictive aspects contribute implicit structures of the target domain and maintain useful domain integrity, descriptive aspects can transcend these limitations for reasoning about the domain, where the aspects to be reasoned about are individually explicated.

Accordingly, the strengths of the respective approaches will depend on the precise tasks to be solved: do we need the representation to identify and understand the essence of the problem? Do we want to solve a specific spatial problem or are we looking for the solution of a class of problems? How much effort is needed to generalize from a specific spatial solution or to apply a general abstract solution to a specific spatial situation?

Another aspect to consider is an issue of scaling. Diagrams are used to support thinking and reasoning in the mind of an observer. They extend our limited imagery abilities that are restricted by the human working memory capacity, which only allows for dealing with a handful of entities at a time (Miller, 1956; Cowan, 2001). In diagrams, this limitation is overcome by providing perceptual access to externally represented spatially organized knowledge. But the bandwidths of perception channels also are limited. Thus, the system consisting of a diagram and an observer is limited by the perceptual field accessible to the observers and by their capacity for imagery. But if we combine perception abilities with action abilities such as eye movement, locomotion, and zooming in and out, external spatial media can provide nearly unlimited information about spatial and other feature dimensions. Local and global spatial relations are capable of integrating knowledge of various kinds.

Thus, although mental imagery and spatial perception are focused and limited, there seems to be no principal spatial limit to exploiting 'knowledge in the world' (Norman 1993) for spatial problem solving and reasoning. The integration of a variety of aspects in the structure of the same medium appears to be a key to identifying ways for solving spatial problems.

Formal representations, on the other hand, serve to analytically disintegrate and linearize the various aspects of knowledge contributing to a problem; they are particularly useful to express and follow-up on thoughts and reasoning by an observer and to describe approaches to spatial problem solving once they have been understood.

In summary: different representations have different strengths. However, these are not systematically exploited. Using the domain of spatial problem solving, we propose to analyze not only the kind of knowledge that is to be dealt with, but to consider also the form in which the problem is originally given, the form in which the solution is needed, and the obstacles that need to be overcome in the various phases of the problem solving process, before deciding on suitable tools for the various phases.

After the introduction to the strings and pins domain and the discussion on spatial instantiations and formal representations of problems, we will now return to the original motivating example of the shortest path problem.

# 5. The shortest path problem

The shortest path problem addresses the problem of identifying the shortest path between two specified nodes in a connected route network. For example, given an abstract route network as depicted in Fig. 9, the task is to identify a shortest path between the two nodes marked by big red dots. There are a variety of ways to solve this problem. Some possibilities are: to use a compass, apply computational algorithms, or exploit physical properties of the representation medium. In the following, we address each possibility in turn, in order to provide the general idea of the approach.



*Figure 9.* Abstract route network. The spatial problem to solve is to identify a shortest path between the two big red points.

To address the question using a compass, one would fix the distance between a pair of compasses and measure the length of each possible path between the two nodes by counting the compass-half-turns needed traverse all line segments related to a path (see Fig. 10a for a depiction of the procedure). To use a computer to solve the problem, the representation has to be transformed into machine-readable form. Then an algorithm has to be applied to compute the solution, e.g., Dijkstra's algorithm. This solution is returned in form of a new representation (e.g., as presented in Fig. 10b).



*Figure 10.* a) Example of how a compass can be used to measure distances in a network; b) Result of applying Dijkstra's Algorithm.

We can also create a strings and pins approach representation of the shortest path problem (as in Fig. 11a), by fixing on the board with pins differently coloured strings which represent the various routes in the network. In order to solve the problem, and therefore identify the shortest path between the two red points, we detach the pins from the board and pull the corresponding two nodes apart, as demonstrated in Fig. 11b).



*Figure 11.* a) Abstract physical string representation of the connected route network; b) Result of moving start and end nodes away from one another. The resulting straight line is the shortest path.

As long as we preserve specific aspects (here the scale of the network segments) of the original problem, all approaches generate a representation of the solution. However, the resulting representations do not always express the solution in an explicit way. For example, the representations of the compass-half-turn counts and the Dijkstra approach yield a quantity, either for each node or for each possible path. These quantities must be compared to derive the final solution, i.e. the length of the shortest path, if it exists. On the other hand, the string approach directly yields a shortest path in form of a straight route of the respective length.

It has to be noted, that in all the addressed approaches only the length of the shortest path is preserved in the solution. That is, the shape and orientation of the original route are lost in the representation of the solution.

# 6. Conclusion: Epistemological implications and cognitive relevance

In order to identify a larger class of spatial problems that we could investigate regarding their approachability with physical problem solving methods or metaphors, we looked at the well-established and famous straightedge and compass constructions in geometry. We provisionally determined that any problem we can solve with straightedge and compass constructions we also should be able to solve with strings and pins constructions: the straightedge is equivalent to a string that has been pulled straight and the compass is equivalent to a straight string segment whose one end is fixed at a given center position and whose other end corresponds to the distance of the radius from the center, i.e. the boundary line of the circle drawn by a compass. Thus, we use different physical metaphors for the abstract notions that Euclid employs; these metaphors enlarge the scope of operations and interpretations.

We have reasons to believe that we may be able to extend Euclidean constructions in interesting ways that allow us to derive additional spatial relations relevant for everyday problem solving, such as the shortest route problem: with Euclidean geometry we are restricted to polygonal and circular route shapes; with strings we gain a potential to determine the length of arbitrary linear shapes – even shapes that may be difficult to characterize formally. For example, we can map linear boundaries of arbitrary convex objects onto strings by pulling them around these objects; then, we can determine their length by pulling the strings straight and determining the distance between the ends of the corresponding string segment; this approach exploits an equivalent spatial property of differently shaped objects. This is a similar approach as Archimedes is said to have used to determine the relative purity of his king's gold crown after having an insight regarding the problem when taking a bath in his bathtub (Vitruvius Pollio 1914).

Whereas Euclidean geometry (just like other formal approaches) is not concerned with the question of how to get spatial relations from the real world into formal systems, we investigate ways in which we could extend the strictly abstract approaches of Euclidean geometry in this direction. So far, the task of relating a representation to the real-world problem is left to human interpreters. We investigate whether we can extend this authority to a larger set of cognitive agents, which includes artificial cognitive agents with perception and action capabilities under very strict conditions. In particular, we only want to permit qualitative checks for *perceived* 

equality or *greater than* relations for entities in the represented environment, like in the postulated Platonic constructions of Euclidean geometry that act exclusively on the conceptual level.

The cognitive relevance of our approach lies in the need to include *knowledge in the world* as an integral part of cognitive systems, just as natural cognitive agents do it. We envision a future where the cost of physical robot operations in the world will be very small; then robots will be able to explore space (as we humans have done since childhood) and solve everyday spatial problems. Seeing the knowledge in the world and acting directly on spatial affordances, rather than formalizing everything to compute formal results, may be able to provide shortcuts which could decide on the tractability of a problem.

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